

A STRUCTURAL THEORY OF COOPERATIVE GAMES WITH NONLINEAR RETURNS

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Classical cooperative game theory takes the characteristic function $v: 2^N \rightarrow \mathbb{R}$ as an unstructured primitive: it can allocate a surplus but cannot derive one. Coalition Intelligence Theory is a structural theory of heterogeneous interacting agents that equips v with decomposable causal factors: the companion paper constructs $v(F) = D(F) \cdot R(F)$ —rank diversity times an aggregate resource function—and proves a Spectral Invariance Theorem separating a structural layer from an interchangeable domain layer. This paper develops the domain layer through five nonlinear specializations, each entering a major field of economics and inheriting the structural theory without re-derivation. Increasing returns to variety nests Ricardo, Heckscher–Ohlin, and Dixit–Stiglitz in one gains decomposition whose dominant margin is determined by rank alone, yields unconditional superadditivity, and reveals that the scarcity rent dissipating as $1/n$ under linear returns converges to a permanent share under power-law returns. Convexity fails universally for every variety premium; full linear independence restores it and places the Shapley value in the core. Network-enhanced returns derive winner-take-all from the convexity of total network value and yield a scale-free curation rule. Coordination-dependent returns produce a three-factor value function whose ceiling-and-floor architecture separates organized firms from decentralized markets, proves superadditivity under capacity non-degradation with a merger-value converse, and provides microfoundations for the theory of the firm without behavioral assumptions. Concave returns produce a sharp diversity threshold for positive gains, and sector-heterogeneous returns extend the decomposition with heterogeneous elasticities. The decomposition transforms cooperative game theory from a theory of division into a theory of design.

KEYWORDS: coalition theory, collective intelligence, cooperative games, Shapley value, increasing returns, network effects, variety premium, core stability, spectral invariance, characteristic function, theory of the firm, coordination capacity.

August 15, 2026

1. INTRODUCTION

1.1. *The Problem*

Cooperative game theory, as formalized by [Shapley \(1953\)](#) and developed through [Bondareva \(1963\)](#), [Shapley \(1967, 1971\)](#), and the Shapley–Ichiishi theorem ([Ichiishi, 1981](#)), is a theory of division. Given a characteristic function $v: 2^N \rightarrow \mathbb{R}$ assigning a value to every coalition, the theory determines how to divide the value fairly (the Shapley value), whether the grand coalition is stable (the core and balancedness), and when the fair allocation is also stable (convexity). These are deep results, and the seven decades since Shapley’s contribution have produced a complete theory of allocation and stability for transferable-utility games.

But the theory takes v as a primitive—an unstructured exogenous table of 2^n numbers. It provides no account of where v comes from: what structural properties of agents generate a particular v , what makes some coalitions more valuable than others, or how to construct the coalition that produces the largest pie. The allocation problem—divide the given pie—is

solved; the *design* problem—build the coalition that produces the largest pie—is not addressed, because v has no internal structure to exploit.

This is a striking state of affairs. The Shapley value tells you what each agent is worth but not *why*—not which structural feature of the agent generates the value. The core tells you whether the grand coalition is stable but not *what* makes it stable. The theory can divide a surplus; it cannot tell you where the surplus comes from, or whether it exists at all.

1.2. Conceptual Foundation

Coalition Intelligence Theory is a structural theory of heterogeneous interacting agents that equips the characteristic function of cooperative games with explicit, decomposable causal factors. The foundational companion (Olfati-Saber, 2026a) constructs the characteristic function from agent primitives: capability vectors factored into magnitudes and profiles, a profile density matrix whose spectrum carries the coalition’s knowledge state, and the factored value $v(F) = D(F) \cdot R(F)$, rank diversity times an aggregate resource function. A Spectral Invariance Theorem separates a structural layer—density matrix, entropy, null space, collapse threshold, adaptive capacity—that is invariant to the choice of R , from a domain layer, carried by R , that is field-specific and interchangeable. One direction of dependence, never the reverse.

That companion develops the linear case $R(F) = \sum_{i \in F} r_i$ and proves the Superadditivity Theorem, the exact Shapley decomposition with its generic-position closed forms, the vulnerability apparatus with its quantitative entropy control, and the Fisher–Coalition adaptive-capacity theorem. But the linear specification—under which all gains arise from profile diversity—assigns zero gains to coalitions of identical agents, and cannot accommodate increasing returns, network effects, or diminishing returns to scale. This paper, the second of the foundational pair, develops the domain layer: five nonlinear specializations of R , each entering a major field of economics, each inheriting the complete structural theory by Spectral Invariance, and each producing domain results—gains decompositions, Shapley comparisons, superadditivity conditions, core stability—that are derived, not assumed. The five cases are not five separate theories but five projections of a single structural theory onto different economic domains.

1.3. Contributions

Case I: Increasing returns to variety ($R = h(n_F) \cdot \sum r_i$, Section 4). Under Dixit–Stiglitz preferences, the gains from coalition formation decompose additively into a classical component driven by profile diversity and a variety component driven by coalition size (Theorem 4.5), and which margin dominates is determined by rank alone (Corollary 4.6): the classical margin dominates whenever $D \geq 2$, at every scale. The game is strictly superadditive for all disjoint coalitions without any condition on profiles (Theorem 4.13), so this single value function nests Ricardo (1817), Heckscher–Ohlin, Krugman (1979), and the Dixit–Stiglitz preference foundation in one gains decomposition. The generic-position Shapley closed forms of the companion extend to any size premium, with the efficiency identity $\sum_i \phi_i = \min(n, m) h(n) R$ recovered analytically (Proposition 4.11)—and they reveal a reversal: the cross-agent rent, which dissipates as $1/n$ under linear returns, converges under the premium $h(n) = n^\alpha$ to the permanent share $\alpha/(1 + \alpha)$ of total value (Corollary 4.12). Increasing returns convert a vanishing scarcity rent into a permanent one. On stability, the failure of convexity is *universal*: for every variety premium there exist configurations violating convexity, by an exact violation formula whose coefficient axiom (V3) forces to be at least one (Theorem 4.15)—no choice of premium can repair the anti-monotone span-expansion indicator—while full linear independence restores convexity and places the Shapley value in the core (Theorem 4.16).

Case II: Network-enhanced returns ($R = g(n_F) \cdot \sum r_i$, Section 5). Among networks of equal size and resources, higher profile diversity is strictly more valuable (Lemma 5.3). The winner-take-all dynamic is located precisely: what matters is the convexity of *total network value* $\chi_g(n) = n g(n)$, not of the scaling factor g alone—marginal user value increases even under concave per-resource scaling whenever χ_g is convex (Proposition 5.5). The premium for span expansion is exactly $g(n_S + 1)(R_S + r_i)$, growing with network size (Proposition 5.6): size and diversity are complements, not substitutes. And the optimal curation rule is scale-free—independent of g entirely (Proposition 5.7).

Case III: Coordination-dependent returns ($R = \mathcal{C}(F) \cdot \sum r_i$, Section 6). The coordination capacity of the coalition’s interaction graph enters as a third explicit factor, giving $v(F) = D(F) \cdot \mathcal{C}(F) \cdot \sum r_i$. A weakest-link ceiling bounds coordination by the ratio of minimum to average degree (Lemma 6.2), and an organizational floor, derived from the discrete Cheeger inequality, guarantees size-independent coordination for graphs satisfying expansion and bounded degree-spread conditions (Theorem 6.4). Superadditivity holds under capacity non-degradation, with an exact decomposition of the merger surplus into capacity-amplification and diversity cross terms; the converse shows that merging can destroy value when post-merger coordination collapses (Theorem 6.5). The entire value differential between two governance structures over the same agents is a single computable capacity ratio (Corollary 6.6). A sufficient condition for firm formation— $D \cdot \theta^2 / 4\psi > 1$, stated entirely in profile geometry and network structure—requires no behavioral assumptions: no opportunism, no bounded rationality, no contract incompleteness, and no market friction.

Case IV: Concave returns to scale ($R = (\sum r_i)^{\gamma_c}$, Section 7). The sharp condition $D(F) > n_F^{1-\gamma_c}$ (Theorem 7.3) determines exactly when rank diversity overcomes the concavity penalty; homogeneous coalitions always suffer net losses under concavity.

Case V: Sector-heterogeneous returns ($R = \sum_\ell h_\ell(n_{F,\ell}) \cdot R_\ell$, Section 8). The gains decomposition extends sector-by-sector with heterogeneous elasticities (Theorem 8.2), capturing the empirical reality that different industries have different returns to variety.

Beyond the five cases, a growth decomposition (Section 9) separates the two channels through which diversity drives output growth—smooth capability accumulation and discrete rank jumps—and an extractive-institutions result (Section 10) shows that the collapse chain of the companion holds under every returns regime, with nonlinear returns making extraction more tempting precisely where its structural consequences are most costly.

1.4. Organization

Section 2 recapitulates the structural layer. Section 3 introduces the generalized domain layer and establishes the general marginal contribution and the pairwise comparison identity. Sections 4–8 develop the five cases. Section 9 derives the growth decomposition and Section 10 the invariance of extractive fragility to R . Section 11 makes the two-layer architecture precise. Section 12 re-examines the traditions the theory touches and states falsifiable predictions; Section 13 concludes.

2. PRIMITIVES AND DEFINITIONS

This section recapitulates the primitives and structural results of the foundational companion (Olfati-Saber, 2026a). All results in this section are proved there and are invariant to the choice of the value function.

2.1. Agents and Capabilities

An agent $i \in A = \{1, \dots, n\}$ has capability vector $\mathbf{q}_i \in \mathbb{R}_{++}^m$, decomposed as $\mathbf{q}_i = r_i \cdot \mathbf{s}_i$, where $r_i = \|\mathbf{q}_i\| > 0$ is the capability magnitude and $\mathbf{s}_i = \mathbf{q}_i/r_i \in \mathbb{S}^{m-1}$ is the capability profile. A coalition $F \subseteq A$ has size $n_F = |F|$, profile matrix $S_F = [\mathbf{s}_i]_{i \in F}$, and rank diversity $D(F) = \text{rank}(S_F)$, with the conventions $\text{span}(\emptyset) = \{0\}$ and $D(\emptyset) = 0$.

2.2. Density Matrix and Diversity

The profile density matrix $\rho_F = \frac{1}{n_F} \sum_{i \in F} \mathbf{s}_i \mathbf{s}_i^\top$ is positive semidefinite with $\text{tr}(\rho_F) = 1$ and $\text{rank}(\rho_F) = D(F)$. Its von Neumann entropy $S(F) = -\text{tr}(\rho_F \log \rho_F)$ measures diversity, and the effective diversity is $d_{\text{eff}}(F) = e^{S(F)} \in [1, \min(n_F, m)]$.

2.3. Coordination and Collective Intelligence

Coordination capacity $\mathcal{C}(F) = \lambda_2(L)/\lambda_2^{\max} \in [0, 1]$ measures network connectivity, where L is the Laplacian of the interaction graph $\mathcal{G}$, $\lambda_2(L)$ its Fiedler eigenvalue, and $\lambda_2^{\max} = \delta(\mathcal{G})/(n_F - 1)$ for aggregate degree $\delta(\mathcal{G})$. Signal fidelity $\mathcal{F}(F) = 1/(1 + A) \in (0, 1]$ measures information quality. Collective intelligence is

$$CI(F) = e^{S(F)} \cdot \mathcal{C}(F) \cdot \mathcal{F}(F), \quad (2.1)$$

and each factor is a bottleneck: the absence of any one caps or destroys CI (Olfati-Saber, 2026a, Definition 2.10 and the surrounding discussion).

2.4. Vulnerability and Resilience

The capability matrix $C_F = \sum_{i \in F} r_i \mathbf{s}_i \mathbf{s}_i^\top$ has null space $\ker(C_F) = \ker(\rho_F)$ of dimension $m - D(F)$. The collapse threshold $\xi_{\text{crit}} = \lambda_{\min}(C_F)$ satisfies: $\xi_{\text{crit}} = 0$ iff $D(F) < m$; and near homogeneity the threshold is controlled linearly by the entropy, $\xi_{\text{crit}} \leq r_{\max} n_F S(F)/((m - 1) \log 2)$ for $S(F) \leq \log 2$, so that along fixed-size sequences with bounded magnitudes, $S(F) \rightarrow 0$ forces $\xi_{\text{crit}} \rightarrow 0$ (Olfati-Saber, 2026a, Theorem 5.3).

2.5. Adaptive Capacity

The Fisher–Coalition Theorem (Olfati-Saber, 2026a, Theorem 5.5) gives

$$\mathbb{E}_\xi[\Delta \bar{w}] = \frac{\eta \kappa}{m} \cdot \sigma_{\text{eff}}^2(F), \quad (2.2)$$

with $\sigma_{\text{eff}}^2(F) = 0$ if and only if $D(F) = 1$: homogeneous coalitions have zero expected adaptive improvement of fitness.

3. THE GENERALIZED DOMAIN LAYER

The structural layer— $\rho_F, D(F), S(F), \xi_{\text{crit}}, \sigma_{\text{eff}}^2$ —is fully determined by profiles and magnitudes, and is invariant to R . The domain layer is where the unstructured v becomes transparent: replacing it with the factored $v = D \cdot R$ allows the economic content to be specified axiomatically, domain by domain, while inheriting the full structural theory.

3.1. The Generalized Value Function

DEFINITION 3.1—Aggregate Resource Function: A map $R: 2^A \rightarrow \mathbb{R}_+$ with $R(\emptyset) = 0$, $R(F) \geq 0$ for all F , and $R(\{i\}) > 0$ for every singleton. (Allowing $R(F) = 0$ for some non-singleton coalitions accommodates returns regimes where disconnected coalitions generate no value.)

DEFINITION 3.2—Generalized Coalition Value Function: The coalition value function is

$$v(F) = D(F) \cdot R(F), \quad (3.1)$$

where $D(F) = \text{rank}(S_F)$ is rank diversity. The conventions $D(\emptyset) = 0$ and $R(\emptyset) = 0$ give $v(\emptyset) = 0$.

The foundational companion develops the linear case $R(F) = \sum_{i \in F} r_i$; the present paper develops five nonlinear cases.

3.2. Spectral Invariance

The Spectral Invariance Theorem (Olfati-Saber, 2026a, Theorem 5.6) establishes that the profile density matrix, von Neumann entropy, effective and rank diversity, capability matrix, null space, collapse threshold, correlated exposure, and Fisher–Coalition adaptive capacity depend only on profiles $\{s_i\}$ and magnitudes $\{r_i\}$, and are invariant to the choice of R . By that theorem, every structural result recapitulated in Section 2 holds for all five nonlinear cases developed below. The domain-layer results of the companion—marginal contributions, super-additivity, the Shapley decomposition and its closed forms—are specific to the linear R and must be re-derived case by case; that is the task of the present paper.

Spectral Invariance is the key: it guarantees that the structural layer—diversity, vulnerability, adaptive capacity—is domain-independent. Each specialization of R below alters the domain layer only, changing how much value the coalition creates and how fairly it is divided, but never how diverse, how fragile, or how adaptive it is.

3.3. General Marginal Contribution

PROPOSITION 3.3—Marginal Contribution under General R : Agent i 's marginal contribution to coalition $S \subseteq A \setminus \{i\}$ (including $S = \emptyset$) under $v(F) = D(F) \cdot R(F)$ is:

Case 1: $s_i \notin \text{span}(S)$:

$$v(S \cup \{i\}) - v(S) = (D_S + 1) \cdot R(S \cup \{i\}) - D_S \cdot R(S). \quad (3.2)$$

Case 2: $s_i \in \text{span}(S)$:

$$v(S \cup \{i\}) - v(S) = D_S \cdot [R(S \cup \{i\}) - R(S)]. \quad (3.3)$$

PROOF: In Case 1, $D(S \cup \{i\}) = D_S + 1$ by the Coverage Theorem (Olfati-Saber, 2026a, Theorem 3.3), whose dichotomy covers $S = \emptyset$ under the conventions of Section 2.1. In Case 2, $D(S \cup \{i\}) = D_S$. Substituting into $v(S \cup \{i\}) - v(S) = D(S \cup \{i\})R(S \cup \{i\}) - D(S)R(S)$ yields both formulas. Q.E.D.

The marginal contribution always decomposes into a diversity bonus (gain from expanding the span, Case 1 only) and a resource bonus (gain from additional resources, amplified by existing diversity). This two-part structure is the mechanism through which the decomposition $v = D \cdot R$ makes the characteristic function transparent: the Shapley value inherits the factorization, and every agent's worth is traceable to its span-expansion contribution and its resource contribution. The relative importance of the two depends on R .

LEMMA 3.4—Gains from Coalition Formation: *Since $v(\{i\}) = 1 \cdot R(\{i\}) = R(\{i\})$, the gains from forming coalition F relative to autarky are*

$$G(F) = D(F) \cdot R(F) - \sum_{i \in F} R(\{i\}). \quad (3.4)$$

PROOF: Direct substitution.

Q.E.D.

3.4. A Pairwise Comparison Identity

The following identity, expressing the difference between two agents' Shapley values as a weighted average of head-to-head value comparisons, is used repeatedly below. It is classical—derivable from the balanced-contributions property of the Shapley value—and we include the short proof for completeness.

LEMMA 3.5—Pairwise Difference Identity: *For any cooperative game v on F with $|F| = n$ and any $i, j \in F$,*

$$\phi_i^{\text{Sh}} - \phi_j^{\text{Sh}} = \sum_{S \subseteq F \setminus \{i, j\}} \frac{|S|!(n - |S| - 2)!}{(n - 1)!} [v(S \cup \{i\}) - v(S \cup \{j\})]. \quad (3.5)$$

PROOF: Partition the subsets in the Shapley formula for ϕ_i according to whether they contain j : every $S' \subseteq F \setminus \{i\}$ is either S or $S \cup \{j\}$ for a unique $S \subseteq F \setminus \{i, j\}$. Writing $w_k = k!(n - k - 1)!/n!$,

$$\begin{aligned} \phi_i - \phi_j &= \sum_{S \subseteq F \setminus \{i, j\}} \left(w_{|S|} [v(S \cup \{i\}) - v(S \cup \{j\})] \right. \\ &\quad \left. + w_{|S|+1} [v(S \cup \{j\} \cup \{i\}) - v(S \cup \{j\})] - w_{|S|+1} [v(S \cup \{i\} \cup \{j\}) - v(S \cup \{i\})] \right). \end{aligned} \quad (3.6)$$

The terms involving $v(S \cup \{i, j\})$ cancel, leaving the coefficient $(w_{|S|} + w_{|S|+1})$ on $v(S \cup \{i\}) - v(S \cup \{j\})$. Finally,

$$w_k + w_{k+1} = \frac{k!(n - k - 2)!}{n!} [(n - k - 1) + (k + 1)] = \frac{k!(n - k - 2)!}{(n - 1)!}. \quad \text{Q.E.D.}$$

4. CASE I: INCREASING RETURNS TO VARIETY

The first case demonstrates that the decomposition $v = D \cdot h \cdot R$ simultaneously nests the classical and modern traditions of international trade theory in a single cooperative game.

The variety premium $h(n_F)$ —derived from Dixit–Stiglitz preferences—captures the insight of [Krugman \(1979\)](#) that even identical economies gain from trade through variety, while the rank-diversity factor $D(F)$ captures the Ricardian and Heckscher–Ohlin insight that different comparative advantages are the source of classical gains. The gains decomposition makes explicit what the synthesis of [Helpman and Krugman \(1985\)](#) left implicit: the two channels interact asymmetrically, and which dominates is determined by rank alone.

4.1. The Variety Premium

DEFINITION 4.1—Variety Premium: A function $h: \mathbb{N} \rightarrow \mathbb{R}_+$ is a *variety premium* if:

(V1) *Normalization*: $h(1) = 1$.

(V2) *Monotonicity*: $h(n+1) > h(n)$ for all $n \geq 1$.

(V3) *Diminishing marginal variety*: $h(n+1) - h(n) \leq h(n) - h(n-1)$ for all $n \geq 2$.

Note that (V1)–(V2) imply $h(n) \geq 1$ for all n , with strict inequality for $n \geq 2$.

DEFINITION 4.2—Value Function with Variety Premium: $R(F) = h(n_F) \cdot \sum_{i \in F} r_i$, yielding

$$v(F) = D(F) \cdot h(n_F) \cdot \sum_{i \in F} r_i. \quad (4.1)$$

When $h \equiv 1$, this reduces to the linear case of the foundational companion.

4.2. Dixit–Stiglitz Microfoundation

PROPOSITION 4.3—Dixit–Stiglitz Variety Premium: *Under CES preferences with elasticity of substitution $\sigma > 1$,*

$$U = \left(\sum_{k=1}^n x_k^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)}, \quad (4.2)$$

the variety premium $h(n) = n^{1/(\sigma-1)}$ satisfies (V1)–(V3) if and only if $\sigma \geq 2$.

PROOF: With symmetric varieties and equal prices, optimal consumption is $x_k = E/(np)$ where E is total expenditure and p the common price, and indirect utility is $U^* = (E/p) \cdot n^{1/(\sigma-1)}$. The ratio $U^*(n)/U^*(1) = n^{1/(\sigma-1)}$ gives $h(n)$. (V1): $h(1) = 1$. (V2): h is strictly increasing since $\alpha := 1/(\sigma-1) > 0$. (V3): $h(n) = n^\alpha$ has non-increasing increments iff $\alpha \leq 1$, i.e. $\sigma \geq 2$; at $\sigma = 2$, $h(n) = n$ is linear and (V3) holds with equality. For $\sigma < 2$, $\alpha > 1$ makes the increments strictly increasing, violating (V3). *Q.E.D.*

REMARK 4.4: The restriction $\sigma \geq 2$ is empirically mild: [Broda and Weinstein \(2006\)](#) report median elasticity estimates near $\sigma = 4$, giving $\alpha = 1/3$ and $h(10) \approx 2.15$.

4.3. Gains Decomposition

THEOREM 4.5—Three-Factor Gains Decomposition: *Under $v(F) = D(F) \cdot h(n_F) \cdot R$ with $R = \sum_{i \in F} r_i$, the gains from trade decompose as*

$$G(F) = \underbrace{(D(F) - 1) \cdot h(n_F) \cdot R}_{\text{classical component}} + \underbrace{(h(n_F) - 1) \cdot R}_{\text{variety component}}. \quad (4.3)$$

- (a) The classical component is positive iff $D(F) \geq 2$ (profile diversity).
- (b) The variety component is positive iff $n_F \geq 2$ (coalition size).
- (c) $G(F) > 0$ whenever $n_F \geq 2$, regardless of profile diversity.

PROOF: By Lemma 3.4 with $R(\{i\}) = h(1)r_i = r_i$, $G(F) = [D(F)h(n_F) - 1] \cdot R$. Write $D(F)h(n_F) - 1 = (D(F) - 1)h(n_F) + (h(n_F) - 1)$. Part (a): $(D(F) - 1)h(n_F) > 0$ iff $D(F) \geq 2$. Part (b): $h(n_F) - 1 > 0$ iff $n_F \geq 2$ by (V1)–(V2). Part (c): for $n_F \geq 2$ the variety component is positive and the classical component is non-negative. *Q.E.D.*

COROLLARY 4.6—Rank-Determined Margin Dominance: *The classical component strictly exceeds the variety component if and only if $D(F) > 2 - 1/h(n_F)$. Since $D(F)$ is a positive integer and $h(n_F) \geq 1$:*

- (a) *The variety component strictly dominates if and only if $D(F) = 1$ and $n_F \geq 2$ (when the classical component is zero).*
- (b) *The classical component strictly dominates whenever $D(F) \geq 2$, for every coalition size n_F . In particular, as $n_F \rightarrow \infty$ with $h(n_F) \rightarrow \infty$, the ratio of classical to variety components converges to $D(F) - 1 \geq 1$: the variety margin never overtakes the classical margin at any scale.*

PROOF: $(D(F) - 1)h(n_F) > h(n_F) - 1$ rearranges to $D(F) > 2 - 1/h(n_F) \in [1, 2)$. An integer $D(F)$ exceeds this threshold iff $D(F) \geq 2$; it falls below iff $D(F) = 1$, in which case the classical component vanishes and the variety component is positive precisely when $n_F \geq 2$. The ratio $(D(F) - 1)h(n_F)/(h(n_F) - 1) \rightarrow D(F) - 1$ as $h(n_F) \rightarrow \infty$. *Q.E.D.*

Which margin dominates is a question of rank, not of scale. Trade among partners with distinct comparative advantage ($D \geq 2$; North–South, inter-industry) is always dominated by the classical margin, and increasingly so in absolute terms as coalitions grow, since variety amplifies the classical component multiplicatively. Pure variety gains dominate only for trade among profile-identical economies ($D = 1$; intra-industry trade among similar developed economies), which is exactly the configuration of Krugman (1979). This is a sharp structural prediction: the decomposition $v = D \cdot h \cdot R$ resolves the inter- versus intra-industry question as a matter of rank, where the synthesis of Helpman and Krugman (1985) left it as an empirical question.

4.4. Marginal Contributions

LEMMA 4.7—Marginal Contribution with Variety Premium: *Agent i 's marginal contribution to $S \subseteq A \setminus \{i\}$ is:*

Case 1: $\mathbf{s}_i \notin \text{span}(S)$:

$$v(S \cup \{i\}) - v(S) = (D_S + 1)h(n_S + 1) \cdot r_i + [(D_S + 1)h(n_S + 1) - D_S h(n_S)] \cdot R_S. \quad (4.4)$$

Case 2: $\mathbf{s}_i \in \text{span}(S)$:

$$v(S \cup \{i\}) - v(S) = D_S h(n_S + 1) \cdot r_i + D_S [h(n_S + 1) - h(n_S)] \cdot R_S, \quad (4.5)$$

where $R_S = \sum_{j \in S} r_j$ and $R_\emptyset = 0$.

PROOF: Substitute $R(S) = h(n_S)R_S$ and $R(S \cup \{i\}) = h(n_S + 1)(R_S + r_i)$ into Proposition 3.3, expand, and collect terms. *Q.E.D.*

COROLLARY 4.8—Positive Marginal Contribution: *Under the variety premium, every agent has a strictly positive marginal contribution to every coalition: for non-empty S , $v(S \cup \{i\}) - v(S) > r_i$; for $S = \emptyset$, $v(\{i\}) - v(\emptyset) = r_i$ exactly.*

PROOF: For $S = \emptyset$, Case 1 with $D_\emptyset = 0$ gives $v(\{i\}) = h(1)r_i = r_i$. For non-empty S in Case 2, $D_S \geq 1$, $h(n_S + 1) > h(n_S) \geq 1$, and $R_S > 0$, so the contribution is at least $D_S h(n_S + 1)r_i > r_i$ plus a strictly positive second term. Case 1 contributions exceed Case 2 contributions term by term, since $(D_S + 1)h(n_S + 1) > D_S h(n_S + 1)$ and $(D_S + 1)h(n_S + 1) - D_S h(n_S) > D_S [h(n_S + 1) - h(n_S)]$. Q.E.D.

Even when the members of a trading bloc have similar capabilities and no comparative advantage over one another, every entrant creates value for every existing member, because each brings differentiated varieties into the common market and the variety premium amplifies the entire resource base. The variety channel operates exactly where the diversity channel has nothing to add. Conversely, trade sanctions within a bloc destroy value on both sides: the bloc loses one variety source, the excluded party loses all of them.

4.5. Shapley Value

THEOREM 4.9—Shapley Value with Variety Premium: *Under $v(F) = D(F)h(n_F) \sum r_i$ with $n_F \geq 2$:*

- (a) *Every agent's Shapley value strictly exceeds its autarky value: $\phi_i^{\text{Sh}} > r_i$ for all $i \in F$.*
- (b) *Holding all profiles and all other magnitudes fixed, ϕ_i^{Sh} is strictly increasing in r_i .*
- (c) (Spanwise dominance.) *Let $i, j \in F$ have equal magnitudes $r_i = r_j = r$, and write $I_k(S) = \mathbf{1}[\mathbf{s}_k \notin \text{span}(S)]$. If $I_i(S) \geq I_j(S)$ for every $S \subseteq F \setminus \{i, j\}$ —that is, if i 's profile expands the span of every subcoalition whose span j 's profile expands—then $\phi_i^{\text{Sh}} \geq \phi_j^{\text{Sh}}$, with strict inequality if $I_i(S) > I_j(S)$ for at least one S .*

PROOF: (a) By Corollary 4.8, in every ordering of F the marginal contribution of i is $\geq r_i$, with equality only in the orderings where i is first and strict inequality in all others. Since $n_F \geq 2$, the orderings where i is not first carry positive total weight, so the average strictly exceeds r_i .

(b) By Lemma 4.7, in both cases the marginal contribution is affine in r_i with strictly positive slope $(D_S + 1)h(n_S + 1)$ or $D_S h(n_S + 1)$ for non-empty S , and slope $h(1) = 1$ at $S = \emptyset$; r_i does not enter R_S , the spans, or the weights. Averaging preserves strict monotonicity.

(c) Apply Lemma 3.5. For $S \subseteq F \setminus \{i, j\}$ with $r_i = r_j = r$,

$$\begin{aligned} v(S \cup \{i\}) - v(S \cup \{j\}) &= h(n_S + 1)(R_S + r)[D(S \cup \{i\}) - D(S \cup \{j\})] \\ &= h(n_S + 1)(R_S + r)[I_i(S) - I_j(S)], \end{aligned}$$

using $D(S \cup \{k\}) = D_S + I_k(S)$. Under the hypothesis every bracket is non-negative; the weights in Lemma 3.5 are strictly positive and $h(n_S + 1)(R_S + r) > 0$, so $\phi_i - \phi_j \geq 0$, strictly if some bracket is strictly positive. Q.E.D.

REMARK 4.10: Part (a) is the key difference from the linear case: under the variety premium, every agent strictly gains from joining the grand coalition, reflecting the insight of [Krugman \(1979\)](#) that even redundant agents create value through variety. Part (c) makes precise the sense in which unique profiles are rewarded: the hypothesis $I_i \geq I_j$ pointwise says that j is redundant to every subcoalition to which i is redundant. Whether frequency dominance $\mathbb{E}_\pi[I_i(S_\pi)] \geq \mathbb{E}_\pi[I_j(S_\pi)]$ suffices without pointwise dominance is an open problem.

In generic position the Shapley value admits closed forms for any size premium, extending the linear-case forms of the companion (Olfati-Saber, 2026a, Proposition 4.7).

PROPOSITION 4.11—Generic-Position Shapley Values under a Size Premium: *Suppose the profiles of F are in generic position (every subset of at most m profiles is linearly independent), and let h be any size-scaling factor with $h(1) = 1$ (a variety premium or a network scaling). Write $n = n_F$, $R_F = \sum_j r_j$, and define the total-value sequence*

$$\chi_D(k) := \min(k, m) \cdot h(k), \quad k = 1, \dots, n. \quad (4.6)$$

Then the Shapley value of every agent is $\phi_i^{\text{Sh}} = \gamma^h r_i + \beta_i^h$ with

$$\gamma^h = \frac{1}{n} \sum_{k=1}^n \chi_D(k), \quad \beta_i^h = (R_F - r_i) \frac{(n-1)\chi_D(n) - \sum_{k=1}^{n-1} \chi_D(k)}{n(n-1)}, \quad (4.7)$$

where γ^h is the same for every agent. The forms satisfy the efficiency identity $\sum_i \phi_i^{\text{Sh}} = \chi_D(n) R_F = v(F)$ exactly, and at $h \equiv 1$ they reduce to the linear-case closed forms of the companion.

PROOF: In generic position, $D(S) = \min(|S|, m)$ for every S , so under a uniform ordering with predecessor count $k = |S_\pi|$ uniform on $\{0, \dots, n-1\}$, the marginal contribution of Lemma 4.7 (which holds verbatim for any size scaling) has r_i -coefficient $D(S \cup \{i\})h(k+1) = \min(k+1, m)h(k+1) = \chi_D(k+1)$ and R_S -coefficient $\chi_D(k+1) - \chi_D(k)$, in both cases (when $k \geq m$ the span indicator vanishes and $\chi_D(k+1) - \chi_D(k) = m[h(k+1) - h(k)]$, matching Case 2; when $k \leq m-1$ it matches Case 1). Averaging the r_i -coefficient over k gives $\gamma^h = \frac{1}{n} \sum_{k=0}^{n-1} \chi_D(k+1)$. For the second term, conditional on $|S_\pi| = k$ the predecessor set is a uniform k -subset of $F \setminus \{i\}$, so $\mathbb{E}[R_{S_\pi}] = k(R_F - r_i)/(n-1)$, and by Abel summation

$$\sum_{k=0}^{n-1} k [\chi_D(k+1) - \chi_D(k)] = (n-1)\chi_D(n) - \sum_{k=1}^{n-1} \chi_D(k),$$

giving β_i^h as displayed. Efficiency: $\sum_i \phi_i = \gamma^h R_F + (n-1)R_F \cdot \frac{(n-1)\chi_D(n) - \sum_{k=1}^{n-1} \chi_D(k)}{n(n-1)} = \frac{R_F}{n} [\sum_{k=1}^n \chi_D(k) + (n-1)\chi_D(n) - \sum_{k=1}^{n-1} \chi_D(k)] = \chi_D(n) R_F = D(F)h(n)R_F$. At $h \equiv 1$, $\chi_D(k) = \min(k, m)$, and evaluating the sums recovers $\gamma = (n+1)/2$ for $n \leq m$, $\gamma = m - m(m-1)/(2n)$ for $n \geq m$, and $\beta_i = (R_F - r_i)K(K+1)/(2n(n-1))$ with $K = \min(m-1, n-1)$. *Q.E.D.*

COROLLARY 4.12—Persistence of the Cross-Agent Rent: *In generic position with $n \geq m$, the aggregate cross-agent rent share is*

$$\frac{\sum_i \beta_i^h}{v(F)} = 1 - \frac{\gamma^h}{\chi_D(n)}, \quad (4.8)$$

and its large- n behavior is determined by the premium. Under linear returns ($h \equiv 1$) the share is $(m-1)/(2n) \rightarrow 0$: the rent dissipates. Under the power premium $h(n) = n^\alpha$ with $\alpha > 0$,

the share converges to

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\gamma^h}{\chi_D(n)} \right) = \frac{\alpha}{1 + \alpha} > 0: \quad (4.9)$$

increasing returns convert the vanishing scarcity rent of the linear case into a permanent share of total value.

PROOF: Summing β_i^h over i and dividing by $v = \chi_D(n)R_F$ gives (4.8) by the efficiency computation above. For $h \equiv 1$ and $n \geq m$, $1 - \gamma/m = (m - 1)/(2n)$ by the closed form. For $h(n) = n^\alpha$ and $n \geq m$, $\chi_D(k) = \min(k, m)k^\alpha$, and $\gamma^h = \frac{1}{n} \sum_{k=1}^n \chi_D(k)$ is dominated by the terms $k \geq m$: $\frac{1}{n} \sum_{k=m}^n mk^\alpha = \frac{m}{n} \cdot \frac{n^{1+\alpha}}{1+\alpha} (1 + o(1)) = \frac{mn^\alpha}{1+\alpha} (1 + o(1))$, while $\chi_D(n) = mn^\alpha$. Hence $\gamma^h/\chi_D(n) \rightarrow 1/(1 + \alpha)$. Q.E.D.

The economics of the two corollaries is a genuine reversal. Under linear returns, the premium earned on other agents' resources is a scarcity rent of position: it accrues only in the early, low-rank segments of the random order, and dissipates as $1/n$ once the coalition outgrows its capability dimension. Under an increasing-returns premium, every arrival—span-expanding or not—raises the value of every predecessor's resources through the size factor, and this amplification does not thin out with n : the cross-agent rent stabilizes at the share $\alpha/(1 + \alpha)$ of total value ($1/4$ at the Broda–Weinstein median $\alpha = 1/3$).

4.6. Superadditivity

THEOREM 4.13—Superadditivity with Variety Premium: *Let F, G be disjoint and non-empty. Then:*

- (a) (General strict superadditivity.) $v(F \cup G) > v(F) + v(G)$, with no condition on profiles. In particular, the variety-premium game is strictly superadditive even for coalitions of profile-identical agents, where the linear game is merely additive.
- (b) (Span-disjoint decomposition.) If in addition $\text{span}(F) \cap \text{span}(G) = \{0\}$, then

$$\begin{aligned} v(F \cup G) = v(F) + v(G) + & \underbrace{[D_F h(n) - D_F h(n_F)] R_F + [D_G h(n) - D_G h(n_G)] R_G}_{\text{variety amplification}} \\ & + \underbrace{D_F h(n) R_G + D_G h(n) R_F}_{\text{cross-diversity gains}}, \end{aligned} \quad (4.10)$$

where $n = n_F + n_G$ and every brace is strictly positive.

- (c) For profile-identical agents with equal magnitudes r and $h(n) = n^\alpha$ ($\alpha > 0$), the gain from merging is $v(F \cup G) - v(F) - v(G) = r[(n_F + n_G)^{1+\alpha} - n_F^{1+\alpha} - n_G^{1+\alpha}] > 0$.

PROOF: (a) Write $n = n_F + n_G$ and note $D(F \cup G) \geq \max(D_F, D_G)$. Then

$$\begin{aligned} v(F \cup G) & \geq \max(D_F, D_G) h(n) (R_F + R_G) \\ & \geq D_F h(n) R_F + D_G h(n) R_G > D_F h(n_F) R_F + D_G h(n_G) R_G = v(F) + v(G), \end{aligned}$$

where the last inequality is strict because $h(n) > h(n_F)$ and $h(n) > h(n_G)$ by (V2) (both F and G are non-empty, so n exceeds each of n_F, n_G) and $R_F, R_G > 0$.

(b) Under span-disjointness, $D(F \cup G) = D_F + D_G$ by the Grassmann dimension formula. Expand $v(F \cup G) = (D_F + D_G)h(n)(R_F + R_G)$ and subtract $v(F) + v(G)$; grouping terms yields the display. Positivity: $h(n) > h(n_F), h(n_G)$ by (V2), and all of D_F, D_G, R_F, R_G are strictly positive.

(c) With $D_F = D_G = D(F \cup G) = 1$ and $R_F = n_F r$, $R_G = n_G r$, the condition reads $(n_F + n_G)^{1+\alpha} > n_F^{1+\alpha} + n_G^{1+\alpha}$, which holds for $\alpha > 0$ since $x \mapsto x^{1+\alpha}$ is strictly superadditive on the positive reals (strict convexity with value 0 at 0). *Q.E.D.*

4.7. Core Stability

A cooperative game is *convex* (Shapley, 1971) if $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $S \subseteq T \subseteq A \setminus \{i\}$; convex games have non-empty cores containing the Shapley value. The CIT game with a variety premium is not convex in general, and the failure is universal: no variety premium escapes it.

LEMMA 4.14—Convexity Deficit: *Let $m = 2$ and, for small $\delta > 0$, take strictly positive unit profiles $\mathbf{s}_i \propto (1, 1)$, $\mathbf{s}_j \propto (1, \delta)$, $\mathbf{s}_k \propto (\delta, 1)$, with magnitudes $(r_i, r_j, r_k) = (1, R, 1)$ and $S = \{j\} \subset T = \{j, k\}$. Then*

$$[v(S \cup \{i\}) - v(S)] - [v(T \cup \{i\}) - v(T)] = R[4h(2) - 1 - 2h(3)] + 4[h(2) - h(3)]. \quad (4.11)$$

At equal magnitudes $R = 1$, convexity requires $6h(3) \geq 8h(2) - 1$.

PROOF: In the configuration, $\text{span}\{\mathbf{s}_j, \mathbf{s}_k\} = \mathbb{R}^2 \ni \mathbf{s}_i$ while $\mathbf{s}_i \notin \text{span}\{\mathbf{s}_j\}$, for every $\delta \in (0, 1)$. With magnitudes $(1, R, 1)$: $v(S \cup \{i\}) - v(S) = 2h(2)(R + 1) - R$ and $v(T \cup \{i\}) - v(T) = 2h(3)(R + 2) - 2h(2)(R + 1)$. Subtracting gives (4.11). At $R = 1$ the display reduces to $8h(2) - 1 - 6h(3)$, so convexity requires $6h(3) \geq 8h(2) - 1$. *Q.E.D.*

THEOREM 4.15—Universal Failure of Convexity: *For every variety premium h satisfying (V1)–(V3), there exist profile-and-magnitude configurations for which $v = D \cdot h \cdot R$ is not convex. No choice of variety premium can restore convexity on unrestricted configurations.*

PROOF: By Lemma 4.14, the convexity deficit (4.11) has coefficient $4h(2) - 1 - 2h(3)$ on R . By (V3) at $n = 2$, $h(3) - h(2) \leq h(2) - h(1) = h(2) - 1$, i.e. $h(3) \leq 2h(2) - 1$, so $4h(2) - 1 - 2h(3) \geq 4h(2) - 1 - 2(2h(2) - 1) = 1$. The deficit therefore exceeds $R - 4[h(3) - h(2)]$, which is positive for $R > 4[h(3) - h(2)]$ (a finite threshold since $h(3) > h(2)$ by (V2)). At equal magnitudes, the requirement $6h(3) \geq 8h(2) - 1$ fails for $h \equiv 1$ (it reads $6 \geq 7$) and, for $h(n) = n^\alpha$, for all $\alpha < \bar{\alpha} \approx 0.4775$, where $\bar{\alpha}$ solves $6 \cdot 3^{\bar{\alpha}} = 8 \cdot 2^{\bar{\alpha}} - 1$. *Q.E.D.*

The failure has a structural cause that no choice of R can repair: the span-expansion indicator $I_i(S)$ is *anti-monotone* in S —a diversity bonus available against a small coalition is extinguished once a larger coalition’s span absorbs $\mathbf{s}_i$ —and part (b) shows that magnitude asymmetry lets this absorption dominate any diminishing-returns premium. The rank function D is a matroid rank function and therefore submodular: $D(S \cup T) + D(S \cap T) \leq D(S) + D(T)$, which means diminishing returns to adding agents in terms of rank. The value function $v = D \cdot R$ is the product of a submodular factor and a supermodular one, and the product inherits the anti-monotone direction from D whenever submodularity dominates—which Lemma 4.14 shows always happens for some configuration. Convexity is restored exactly when no agent is ever absorbed:

THEOREM 4.16—Convexity and Core Non-Emptiness under Full Independence: *Suppose the profiles $\{s_i\}_{i \in A}$ are linearly independent (so $D(S) = |S|$ for every $S \subseteq A$, which requires $n_A \leq m$), and suppose $\chi(n) := n \cdot h(n)$ has non-decreasing increments: $\chi(n+1) - \chi(n) \geq \chi(n) - \chi(n-1)$ for $n \geq 1$, with $\chi(0) = 0$. Then $v(F) = D(F)h(n_F) \sum_F r_i$ is convex; consequently the core is non-empty and contains the Shapley value. The increment condition holds for every $h(n) = n^\alpha$ with $\alpha \geq 0$, including the Dixit–Stiglitz premium ($\chi(n) = n^{\sigma/(\sigma-1)}$ is convex).*

PROOF: Under full independence, every marginal contribution falls in Case 1 of Lemma 4.7 with $D_S = n_S$, and since $(n_S + 1)h(n_S + 1) = \chi(n_S + 1)$,

$$v(S \cup \{i\}) - v(S) = \chi(n_S + 1)r_i + [\chi(n_S + 1) - \chi(n_S)]R_S. \quad (4.12)$$

Take $S \subseteq T \subseteq A \setminus \{i\}$. Since h is positive and increasing, χ is strictly increasing, so $\chi(n_T + 1) \geq \chi(n_S + 1)$: the first term is weakly larger at T . By the increment condition, $\chi(n_T + 1) - \chi(n_T) \geq \chi(n_S + 1) - \chi(n_S) \geq 0$, and $R_T \geq R_S \geq 0$: the second term is weakly larger at T . Hence marginal contributions are non-decreasing in the coalition, which is convexity; core non-emptiness and Shapley membership follow from Shapley (1971). For $h(n) = n^\alpha$, $\chi(n) = n^{1+\alpha}$ is convex on $[0, \infty)$, hence has non-decreasing increments on the integers. Q.E.D.

REMARK 4.17—Open Problem: Between the two extremes—full independence (convex, Theorem 4.16) and configurations with absorbable profiles (non-convex, Theorem 4.15)—a complete characterization of the profile geometries for which the CIT game is convex, or for which the core is non-empty despite non-convexity, remains open. Non-convexity does not imply an empty core; settling core non-emptiness in general requires checking the Bondareva–Shapley balancedness conditions for the family $v = D \cdot h \cdot R$, which we leave for future work.

4.8. Unification of Trade Theories

The extended value function nests the foundational theories as special cases, each recoverable by specializing the decomposition $v = D \cdot h \cdot R$.

Ricardo. When $h \equiv 1$ and profiles differ: $G(F) = (D(F) - 1) \cdot R$, with profile non-parallelism necessary and sufficient (Ricardo, 1817).

Heckscher–Ohlin. When $h \equiv 1$ and factor endowments differ: different capital–labor ratios generate non-parallel profiles whenever the technology matrix has rank 2 (Olfati-Saber, 2026a, Propositions 7.1–7.2).

Krugman. When $D(F) = 1$ and h is non-trivial: $G(F) = (h(n_F) - 1) \cdot R$, positive for $n_F \geq 2$ regardless of profile similarity (Krugman, 1979).

The Helpman–Krugman synthesis. When profiles differ and h is non-trivial:

$$G(F) = \underbrace{(D(F) - 1) \cdot h(n_F) \cdot R}_{\text{inter-industry (H-O)}} + \underbrace{(h(n_F) - 1) \cdot R}_{\text{intra-industry (Krugman)}}. \quad (4.13)$$

The two channels interact asymmetrically: classical gains are amplified by variety ($h > 1$), while variety gains are independent of diversity (Corollary 4.6). The decomposition makes transparent what each tradition contributes: Ricardo and Heckscher–Ohlin contribute the diversity factor D ; Krugman and Dixit–Stiglitz contribute the variety factor h ; and the value function $v = D \cdot h \cdot R$ shows that they are not competing theories but complementary factors of a single characteristic function.

5. CASE II: NETWORK-ENHANCED RETURNS

The second case demonstrates that the decomposition $v = D \cdot g \cdot R$ produces results that the standard network-externalities framework—which treats users as homogeneous within each side of the market—does not achieve. The structural insight is that network size and network diversity are complementary, not substitutable: the network scaling $g(n_F)$ amplifies the diversity multiplier $D(F)$ multiplicatively, so that diversity becomes more valuable in absolute terms as the network grows. This provides a micro-foundation for the winner-take-all dynamic that goes beyond demand-side economies of scale: platforms that attract diverse users dominate not only through size but through the multiplied diversity premium.

5.1. Value Function

DEFINITION 5.1—Network Scaling Function: A non-decreasing $g: \mathbb{N} \rightarrow \mathbb{R}_+$ with $g(1) = 1$ captures how network size amplifies value. We set $R(F) = g(n_F) \cdot \sum_{i \in F} r_i$, yielding

$$v(F) = D(F) \cdot g(n_F) \cdot \sum_{i \in F} r_i. \quad (5.1)$$

Different network-effect models correspond to different choices of g (each normalized so that $g(1) = 1$): Sarnoff ($g(n) = n$), Metcalfe ($g(n) = n^2$), Reed ($g(n) = 2^{n-1}$), and sublinear congestion ($g(n) = n^\alpha$, $\alpha < 1$). Unlike a variety premium, g is not required to satisfy (V3); Metcalfe and Reed scaling are convex.

Because Proposition 3.3, Lemma 4.7, and Proposition 4.11 use only $h(1) = 1$ and never (V2)–(V3), they hold verbatim with g in place of h ; in particular the generic-position closed forms and their efficiency identity apply to every network scaling.

5.2. Gains Decomposition

COROLLARY 5.2—Network Gains Decomposition: Under $v(F) = D(F) \cdot g(n_F) \cdot R$ with $R = \sum_{i \in F} r_i$:

$$G(F) = \underbrace{(D(F) - 1) \cdot g(n_F) \cdot R}_{\text{diversity component}} + \underbrace{(g(n_F) - 1) \cdot R}_{\text{network component}}. \quad (5.2)$$

- (a) The diversity component is positive iff $D(F) \geq 2$.
- (b) The network component is positive iff $n_F \geq 2$ and $g(n_F) > 1$.
- (c) The diversity multiplier is bounded by the ambient dimension: $D(F) - 1 \leq m - 1$ for every coalition, whereas the network factor $g(n_F) - 1$ is unbounded in n_F under Metcalfe or Reed scaling. At scale, network effects dominate any fixed diversity advantage.

PROOF: The decomposition and (a)–(b) follow exactly as in Theorem 4.5, with $g(1) = 1$ in place of (V1) and the stated condition $g(n_F) > 1$ in place of (V2). For (c), $D(F) \leq \min(n_F, m) \leq m$ bounds the multiplier, while $g(n) - 1 \rightarrow \infty$ for $g(n) = n^2$ or 2^{n-1} . *Q.E.D.*

5.3. Diversity-Enhanced Network Effects

LEMMA 5.3—Diversity Multiplier: For two networks F and F' with $n_F = n_{F'}$ and $\sum_F r_i = \sum_{F'} r_i$, higher diversity $D(F) > D(F')$ at equal resources yields strictly higher value: $v(F) > v(F')$.

PROOF: $v(F) = D(F)g(n_F)R$ and $v(F') = D(F')g(n_F)R$ with $D(F) > D(F')$ and $g(n_F)R > 0$. Q.E.D.

5.4. Marginal User Value

LEMMA 5.4—Marginal User Contribution: *Under $v(F) = D(F) \cdot g(n_F) \cdot R$:*

Case 1: $s_i \notin \text{span}(S)$:

$$\Delta v_i = (D_S + 1)g(n_S + 1) \cdot r_i + [(D_S + 1)g(n_S + 1) - D_S g(n_S)] \cdot R_S. \quad (5.3)$$

Case 2: $s_i \in \text{span}(S)$:

$$\Delta v_i = D_S g(n_S + 1) \cdot r_i + D_S [g(n_S + 1) - g(n_S)] \cdot R_S. \quad (5.4)$$

PROOF: Substitute $R(S) = g(n_S)R_S$ into Proposition 3.3 and expand. Q.E.D.

The next two results were referenced informally above; we now state them as propositions. The first locates the winner-take-all dynamic precisely: the economically relevant notion of congestion is concavity of *total network value*, not of the per-resource scaling alone.

PROPOSITION 5.5—Homogeneous-Network Marginal Value: *Let all users have identical profiles ($D = 1$) and equal magnitudes r , and write $\chi_g(n) := n \cdot g(n)$ for the total network value function, so that $v(F_n) = \chi_g(n) \cdot r$. Then the marginal value of the $(n + 1)$ -th user is*

$$\Delta v_{n+1} = [\chi_g(n + 1) - \chi_g(n)] \cdot r, \quad (5.5)$$

which is non-increasing in n if and only if χ_g has non-increasing increments. In particular:

- (a) Concavity of g does not suffice: for $g(n) = n^\alpha$ with $\alpha \in (0, 1)$, g is concave yet $\chi_g(n) = n^{1+\alpha}$ is strictly convex, so marginal user value is strictly increasing despite diminishing per-resource returns.
- (b) Diminishing marginal user value requires saturation: for $g(n) = 1 + c(1 - 1/n^2)$ with $c > 0$, $\chi_g(n)$ has strictly decreasing increments.

PROOF: With $D = 1$ throughout, $v(F_n) = 1 \cdot g(n) \cdot nr = \chi_g(n)r$, and (5.5) follows by taking the difference. (a) $\chi_g(n) = n^{1+\alpha}$ has exponent $1 + \alpha > 1$, hence strictly convex with strictly increasing increments: the increment of the scaling factor multiplies an ever-larger resource base. (b) Direct computation: $\chi_g(n + 1) - \chi_g(n) = (1 + c) + c[1/n - 1/(n + 1)] = (1 + c) + c/(n(n + 1))$, strictly decreasing in n ; and $g(1) = 1$. Q.E.D.

The winner-take-all dynamic is therefore driven by the interaction between the scaling factor and the resource base, not by the scaling factor alone: any g for which χ_g is convex—including every $g(n) = n^\alpha$ with $\alpha > 0$, however sublinear—produces increasing returns to the marginal user, and only scalings that saturate fast enough to make χ_g concave produce congestion in the economically relevant sense.

PROPOSITION 5.6—Growing Span-Expansion Premium: *The premium a marginal user earns for expanding the span—the difference between the Case 1 and Case 2 contributions of Lemma 5.4—is exactly*

$$\Delta v_i^{(1)} - \Delta v_i^{(2)} = g(n_S + 1)(R_S + r_i). \quad (5.6)$$

Since g is non-decreasing and R_S is increasing along any growth path of the network, the premium is non-decreasing in n_S and strictly increasing in R_S : under network effects, profile diversity becomes more valuable in absolute terms as the network grows.

PROOF: Subtract (5.4) from (5.3): the r_i -terms differ by $g(n_S + 1)r_i$ and the R_S -terms by $[(D_S + 1)g(n_S + 1) - D_S g(n_S)] - D_S [g(n_S + 1) - g(n_S)] = g(n_S + 1)$ times R_S . *Q.E.D.*

Network size and network diversity are thus complements, not substitutes: the same user is worth strictly more to a larger network precisely when they bring a new capability direction, and the complementarity is priced exactly by (5.6).

PROPOSITION 5.7—Optimal Platform Curation: *If a platform with existing user set F acquires exactly k users from a candidate pool $\mathcal{P}$, the value-maximizing selection satisfies*

$$S^* = \arg \max_{S \subseteq \mathcal{P}, |S|=k} D(F \cup S) \cdot (R_F + R_S), \quad (5.7)$$

independently of the network scaling g . The optimum trades off span expansion against magnitude. In particular, among candidate sets of equal total magnitude, S^ maximizes $D(F \cup S)$: with magnitudes held equal, span-expanding users are strictly preferred to redundant ones.*

PROOF: The full objective is $\max_S D(F \cup S) \cdot g(n_F + k) \cdot (R_F + R_S)$. The factor $g(n_F + k)$ depends only on k , which is fixed, so it is constant across feasible selections and can be dropped. For the final claim, with R_S fixed across candidate sets the objective is proportional to $D(F \cup S)$, and a selection achieving strictly larger rank achieves strictly larger value. *Q.E.D.*

REMARK 5.8: The curation rule is scale-free: the ranking of candidate sets does not depend on g at all, so the same acquisition policy is optimal for a startup and for a platform at Metcalfe scale. Without the equal-magnitude qualification, no general preference for span-expanding users follows: a redundant user of sufficiently large magnitude outranks a span-expanding user of small magnitude. Curation toward diversity is optimal on the margin where magnitudes are comparable. This is a prescriptive result: the decomposition $v = D \cdot g \cdot R$ tells the platform not merely what its network is worth but whom to add next.

6. CASE III: COORDINATION-DEPENDENT RETURNS

Cases I–II modify how coalition *size* enters R . The third case introduces a structurally different channel: how the members of a coalition are *connected to one another*. The instrument is the coordination capacity $\mathcal{C}(F) \in [0, 1]$ already defined in Section 2. Letting coordination enter R produces a three-factor value function whose structure is rich enough to separate efficient organizations from dysfunctional ones, and to provide micro-foundations for the theory of the firm without behavioral assumptions.

6.1. The Coordination Return

DEFINITION 6.1—Coordination-Dependent Value Function: The aggregate resource function is $R(F) = \mathcal{C}(F) \cdot \sum_{i \in F} r_i$, so that the coalition value function is

$$v(F) = D(F) \cdot \mathcal{C}(F) \cdot \sum_{i \in F} r_i. \quad (6.1)$$

The conventions $D(\emptyset) = 0$, $R(\emptyset) = 0$ give $v(\emptyset) = 0$; $D(\{i\}) = 1$, $\mathcal{C}(\{i\}) = 1$ give $v(\{i\}) = r_i$.

The value of a coalition is its capability diversity, times its coordination capacity, times its resource base—three separately measurable factors, each a bottleneck. Diversity and the resource base depend only on the agents' capability vectors; coordination capacity depends only on the interaction graph. By the Spectral Invariance Theorem (Olfati-Saber, 2026a, Theorem 5.6), the structural layer—the density matrix, entropy, effective diversity, null space, collapse threshold, and adaptive capacity—is unchanged under this specification. The function $R(F) = \mathcal{C}(F) \cdot \sum r_i$ satisfies $R(\emptyset) = 0$, $R(\{i\}) = r_i > 0$, and $R(F) \geq 0$ for all F , with $R(F) = 0$ exactly when the interaction graph is disconnected; the relaxation from strict to non-strict positivity is immaterial to the structural theory.

The gains from forming the coalition F are

$$G(F) = v(F) - \sum_{i \in F} v(\{i\}) = (D(F) \cdot \mathcal{C}(F) - 1) \cdot \sum_{i \in F} r_i. \quad (6.2)$$

Formation creates value exactly when $D(F) \cdot \mathcal{C}(F) > 1$: diversity times coordination must exceed the standalone benchmark.

6.2. The Capacity Ceiling: A Weakest-Link Bound

Write $\bar{d} = \delta(\mathcal{G})/n_F$ for the average degree and $d_{\min}$ for the minimum degree of the interaction graph. The following results rely on algebraic graph theory and matrix theory.

LEMMA 6.2—Capacity Ceiling: *For every connected graph $\mathcal{G}$ on $n_F \geq 2$ vertices,*

$$\mathcal{C}(F) \leq \frac{d_{\min}}{\bar{d}}. \quad (6.3)$$

Equality holds for the complete graph K_{n_F} , where $d_{\min} = \bar{d} = n_F - 1$ and $\mathcal{C} = 1$.

PROOF: Let x denote the vector of values of the graph nodes and u be a vertex of minimum degree, and set $x = e_u - \mathbf{1}/n_F$, where e_u is the vector with 1 at coordinate u and zero elsewhere, so that $x \perp \mathbf{1}$. Only edges incident to u contribute to $x^\top Lx$: for an edge $\{u, j\}$, $(x_u - x_j)^2 = 1$; for an edge avoiding u , both entries equal $-1/n_F$ and contribute zero. Hence $x^\top Lx = d_{\min}$. Also $\|x\|^2 = (1 - 1/n_F)^2 + (n_F - 1)/n_F^2 = (n_F - 1)/n_F$. By the Courant–Fischer characterization,

$$\lambda_2 \leq \frac{x^\top Lx}{\|x\|^2} = \frac{n_F d_{\min}}{n_F - 1}.$$

Dividing by $\lambda_2^{\max} = \delta(\mathcal{G})/(n_F - 1)$ gives $\mathcal{C}(F) \leq n_F d_{\min}/\delta(\mathcal{G}) = d_{\min}/\bar{d}$. For K_{n_F} , the Laplacian has spectrum $\{0, n_F^{(n_F-1)}\}$, $\delta = n_F(n_F - 1)$, and $\mathcal{C} = n_F(n_F - 1)/(n_F(n_F - 1)) = 1 = d_{\min}/\bar{d}$. Q.E.D.

Coordination capacity is governed by the worst-connected member: a coalition is only as coordinated as its weakest link. A coalition can have arbitrarily high average degree while a single poorly attached member caps $\mathcal{C}$ for everyone.

6.3. The Organizational Capacity Floor

The ceiling says which graphs coordinate poorly. We now define the class of graphs that coordinate well and prove that membership guarantees a capacity floor independent of coalition size.

DEFINITION 6.3—Organizational Graph: A connected simple graph $\mathcal{G}$ on $n_F \geq 2$ vertices is (θ, ψ) -organized, with $\theta > 0$ and $\psi \geq 1$, if it satisfies two conditions:

- (i) *Expansion*: the isoperimetric number $\iota(\mathcal{G}) = \min_{|S| \leq n_F/2} |\partial S|/|S| \geq \theta \bar{d}$ (the Cheeger constant), where $|\partial S|$ counts edges crossing the cut $(\bar{S}, F \setminus S)$.
- (ii) *Bounded degree spread*: $d_{\max}/\bar{d} \leq \psi$.

Condition (i) requires that no subset is thinly connected to the rest: every subgroup has boundary proportional to the average interaction rate. Condition (ii) prevents coordination from concentrating in overloaded hubs. Together they formalize what it means for a graph to be *deliberately organized* rather than assembled by decentralized attachment.

THEOREM 6.4—Organizational Capacity Floor: *Let $\mathcal{G}$ be (θ, ψ) -organized on $n_F \geq 2$ vertices. Then*

$$\mathcal{C}(F) \geq \frac{\theta^2}{2\psi} \cdot \frac{n_F - 1}{n_F} \geq \frac{\theta^2}{4\psi}. \quad (6.4)$$

The bound does not depend on n_F beyond the factor $(n_F - 1)/n_F \geq 1/2$.

PROOF: The discrete Cheeger inequality (Alon, 1985, Mohar, 1989) states $\iota(\mathcal{G})^2 \leq 2d_{\max} \lambda_2$, hence $\lambda_2 \geq \iota(\mathcal{G})^2/(2d_{\max})$. Substituting (i) in the numerator and (ii) in the denominator,

$$\lambda_2 \geq \frac{\theta^2 \bar{d}^2}{2\psi \bar{d}} = \frac{\theta^2}{2\psi} \bar{d}.$$

Dividing by $\lambda_2^{\max} = \delta(\mathcal{G})/(n_F - 1) = n_F \bar{d}/(n_F - 1)$ yields $\mathcal{C}(F) \geq (\theta^2/2\psi) \cdot (n_F - 1)/n_F$, and $(n_F - 1)/n_F \geq 1/2$ for $n_F \geq 2$. *Q.E.D.*

Three consequences carry the economics. *Size-independence*: the floor does not decay with n_F , so an organized coalition can grow without its coordination collapsing, provided it maintains conditions (i)–(ii). *The boundary*: the coalition stops growing when it can no longer maintain expansion or bounded degree spread—when new members attach thinly or overload existing hubs. *Market separation*: decentralized exchange networks, whose topologies are assembled by random or preferential attachment rather than deliberate design, typically violate condition (i) at the periphery and are bounded by the ceiling alone; organized coalitions enjoy the floor. The gap between the two is the coordination advantage of the firm.

6.4. Superadditivity under Capacity Non-Degradation

THEOREM 6.5—Superadditivity under Capacity Non-Degradation: *Let F, F' be disjoint non-empty coalitions with $\text{span}(F) \cap \text{span}(F') = \{0\}$, and write $D = D(F)$, $D' = D(F')$, $\mathcal{C}_\cup = \mathcal{C}(F \cup F')$. If*

$$\mathcal{C}_\cup \geq \max(\mathcal{C}(F), \mathcal{C}(F')) > 0, \quad (6.5)$$

then $v(F \cup F') > v(F) + v(F')$, with the exact decomposition

$$\begin{aligned} v(F \cup F') - v(F) - v(F') &= \underbrace{D[\mathcal{C}_\cup - \mathcal{C}(F)] R_F + D'[\mathcal{C}_\cup - \mathcal{C}(F')] R_{F'}}_{\text{capacity amplification} \geq 0} \\ &\quad + \underbrace{\mathcal{C}_\cup (D R_{F'} + D' R_F)}_{\text{diversity cross terms} > 0}, \end{aligned} \quad (6.6)$$

where $R_F = \sum_{i \in F} r_i$ and $R_{F'} = \sum_{i \in F'} r_i$. Conversely, the non-degradation condition cannot be dropped: if

$$\mathcal{C}_\cup < \frac{D\mathcal{C}(F)R_F + D'\mathcal{C}(F')R_{F'}}{(D+D')(R_F + R_{F'})}, \quad (6.7)$$

then $v(F \cup F') < v(F) + v(F')$ —merging destroys value.

PROOF: By span-disjointness and the Grassmann dimension formula, $D(F \cup F') = D + D'$. Expanding,

$$\begin{aligned} v(F \cup F') &= (D + D')\mathcal{C}_\cup(R_F + R_{F'}) \\ &= D\mathcal{C}_\cup R_F + D'\mathcal{C}_\cup R_{F'} + \mathcal{C}_\cup(DR_{F'} + D'R_F) \\ &= v(F) + v(F') + D[\mathcal{C}_\cup - \mathcal{C}(F)]R_F + D'[\mathcal{C}_\cup - \mathcal{C}(F')]R_{F'} \\ &\quad + \mathcal{C}_\cup(DR_{F'} + D'R_F), \end{aligned}$$

which is (6.6). Under (6.5), the amplification terms are non-negative and the cross terms are strictly positive since $\mathcal{C}_\cup > 0$, $D, D' \geq 1$, and $R_F, R_{F'} > 0$. For the converse, $v(F \cup F') = (D + D')\mathcal{C}_\cup(R_F + R_{F'})$ compares directly with $v(F) + v(F') = D\mathcal{C}(F)R_F + D'\mathcal{C}(F')R_{F'}$, and (6.7) is exactly the condition $v(F \cup F') < v(F) + v(F')$. Q.E.D.

The theorem differs from the linear-case Superadditivity Theorem (Olfati-Saber, 2026a, Theorem 4.3) in requiring capacity non-degradation, and the converse clause shows the requirement is substantive: two complete teams of span-disjoint agents joined by a single cross-tie have $\mathcal{C}_\cup = O(1/n_F^2)$, and the merger destroys value for every team size $n_F \geq 3$, because one cross-tie collapses the merged capacity by a factor of order n_F^2 . Post-merger value creation requires building cross-connectivity until $\mathcal{C}_\cup$ recovers past the threshold (6.7)—integration, not mere internalization, is what the arithmetic rewards.

6.5. The Spectral Comparison and Formation Condition

COROLLARY 6.6—Value Ratio Equals Capacity Ratio: *Let F be a set of agents and let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two connected interaction graphs on F . Then*

$$\frac{v_1(F)}{v_2(F)} = \frac{\mathcal{C}_1(F)}{\mathcal{C}_2(F)} = \frac{\lambda_2(L_1)}{\lambda_2(L_2)} \cdot \frac{\delta(\mathcal{G}_2)}{\delta(\mathcal{G}_1)}. \quad (6.8)$$

PROOF: The diversity $D(F)$ and the resource base $\sum_{i \in F} r_i$ depend only on the agents' capability vectors, not on the interaction graph. Dividing the value functions under the two graphs cancels the common factors, leaving the capacity ratio; expanding $\mathcal{C}$ by its definition gives the second equality. Q.E.D.

The entire value differential between organizing a set of agents as a firm and leaving them to interact through a decentralized market is a ratio of algebraic connectivities weighted by interaction intensities. No behavioral content enters the comparison.

COROLLARY 6.7—Sufficient Condition for Formation: *Let $\mathcal{G}$ be (θ, ψ) -organized on F with $D(F) \geq 2$. If*

$$D(F) \cdot \frac{\theta^2}{4\psi} > 1, \quad (6.9)$$

then $v(F) > \sum_{i \in F} r_i$: forming the firm is strictly beneficial for the members.

PROOF: The floor (Theorem 6.4) gives $\mathcal{C}(F) \geq \theta^2/4\psi$, so $D(F) \cdot \mathcal{C}(F) \geq D(F) \cdot \theta^2/4\psi > 1$, and (6.2) is positive. Q.E.D.

Condition (6.9) is stated entirely in profile geometry (D) and network structure (θ, ψ). It does not involve a contracting technology, an ownership variable, a monitoring arrangement, or a behavioral disposition. The condition answers the formation question—why firms form—directly from the structural theory: firms form because coalition formation creates value under the coordination return, and the organized graph sustains the coordination that makes the value positive. The boundary of the firm lies where the organized graph can no longer be maintained—where adding members breaks expansion or inflates degree spread—and the Spectral Comparison (Corollary 6.6) locates the governance advantage of the firm over the market in a single computable ratio.

7. CASE IV: CONCAVE RETURNS TO SCALE

The fourth case addresses the natural objection that returns to scale may be diminishing rather than increasing. Under concave returns, the decomposition $v = D \cdot R^{\gamma_c}$ reveals a sharp diversity threshold: homogeneous coalitions always suffer losses from concavity, and only sufficient rank diversity can overcome the penalty. The threshold is prescriptive—it tells the coalition designer exactly how much diversity is needed to make formation worthwhile under diminishing returns.

DEFINITION 7.1: $R(F) = (\sum_{i \in F} r_i)^{\gamma_c}$ for $\gamma_c \in (0, 1)$, which yields the value function $v(F) = D(F) \cdot (\sum_{i \in F} r_i)^{\gamma_c}$.

LEMMA 7.2—Gains under Concave Returns: $G(F) = D(F) \cdot (\sum_i r_i)^{\gamma_c} - \sum_i r_i^{\gamma_c}$.

PROOF: Since $v(\{i\}) = 1 \cdot r_i^{\gamma_c}$, apply Lemma 3.4. Q.E.D.

THEOREM 7.3—Diversity Threshold for Positive Gains: *For agents with equal magnitudes ($r_i = r$) and $n_F \geq 2$, $G(F) > 0$ if and only if $D(F) > n_F^{1-\gamma_c}$. In particular, homogeneous coalitions ($D(F) = 1$) always suffer strict losses from concavity, and the threshold is feasible only when the ambient dimension permits it: positive gains require $m \geq D(F) > n_F^{1-\gamma_c}$.*

PROOF: With $r_i = r$: $G = r^{\gamma_c} (D(F) n_F^{\gamma_c} - n_F)$, positive iff $D(F) > n_F^{1-\gamma_c}$. For $D(F) = 1$ and $n_F \geq 2$: $n_F^{\gamma_c} - n_F < 0$ since $\gamma_c < 1$. Feasibility follows from $D(F) \leq \min(n_F, m)$. Q.E.D.

For $\gamma_c = 1/2$, a coalition of 100 equal-magnitude agents needs at least 11 independent capability directions ($D > \sqrt{100} = 10$, hence $D \geq 11$, requiring $m \geq 11$) to generate positive gains.

8. CASE V: SECTOR-HETEROGENEOUS RETURNS

The fifth case extends the decomposition to multi-sector economies, demonstrating that the gains decomposition generalizes sector-by-sector with heterogeneous elasticities. This captures the empirical reality that different industries have different returns to variety, and connects to the finding of Broda and Weinstein (2006) that the variety-gains channel varies substantially across product categories.

DEFINITION 8.1: Partition activity into m sectors. Each agent i carries sector resources $R_\ell(\{i\}) \geq 0$ with $\sum_\ell R_\ell(\{i\}) > 0$, and is *active* in sector ℓ if $R_\ell(\{i\}) > 0$. For a coalition F , let $n_{F,\ell} = |\{i \in F : R_\ell(\{i\}) > 0\}|$ be the count of active agents in sector ℓ , and assume:

(S1) *Additivity*: $R_\ell(F) = \sum_{i \in F} R_\ell(\{i\})$ for every F and ℓ .

(S2) *Sector variety premia*: each $h_\ell : \mathbb{N} \rightarrow \mathbb{R}_+$ is a variety premium (Definition 4.1), extended by $h_\ell(0) = 1$ (the value of h_ℓ on empty sectors is immaterial since it multiplies $R_\ell = 0$).

The aggregate resource function and value function are

$$R(F) = \sum_{\ell=1}^m h_\ell(n_{F,\ell}) \cdot R_\ell(F), \quad v(F) = D(F) \cdot \sum_{\ell=1}^m h_\ell(n_{F,\ell}) \cdot R_\ell(F). \quad (8.1)$$

THEOREM 8.2—Sector-Heterogeneous Gains: *Under (S1)–(S2), the gains from coalition formation decompose as*

$$G(F) = \underbrace{(D(F) - 1) \cdot \sum_{\ell} h_\ell(n_{F,\ell}) \cdot R_\ell(F)}_{\text{diversity component}} + \underbrace{\sum_{\ell} (h_\ell(n_{F,\ell}) - 1) \cdot R_\ell(F)}_{\text{sector-specific variety components}}. \quad (8.2)$$

The sector- ℓ variety component is positive iff at least two coalition members are active in sector ℓ .

PROOF: For a singleton, $n_{\{i\},\ell} = 1$ in every sector where i is active and the premium there is $h_\ell(1) = 1$, so $v(\{i\}) = \sum_\ell R_\ell(\{i\})$. Summing over $i \in F$ and using (S1), $\sum_i v(\{i\}) = \sum_\ell R_\ell(F)$. Then

$$G(F) = D(F) \sum_{\ell} h_\ell R_\ell - \sum_{\ell} R_\ell = (D(F) - 1) \sum_{\ell} h_\ell R_\ell + \sum_{\ell} (h_\ell - 1) R_\ell,$$

with all arguments $n_{F,\ell}$ suppressed. Positivity of the sector- ℓ term: $h_\ell(n_{F,\ell}) > 1$ iff $n_{F,\ell} \geq 2$, by (V1)–(V2), and $R_\ell(F) > 0$ whenever some member is active in ℓ . Q.E.D.

This captures the empirical reality that industries differ in their elasticities: pharmaceuticals (σ low, steep h_ℓ) benefit more from variety than commodities (σ high, flat h_ℓ), consistent with Broda and Weinstein (2006). The decomposition $v = D \cdot \sum_\ell h_\ell R_\ell$ preserves the structural/domain separation: the diversity factor $D(F)$ multiplies the entire sector-aggregated resource function, so every structural result—null space, collapse threshold, adaptive capacity—is inherited by Spectral Invariance, while the domain layer decomposes sector by sector.

9. A GROWTH DECOMPOSITION

This section derives the implications of the value function for growth accounting. Because the framework is static in its primitives, the dynamic statements below take the form of a decomposition plus an explicitly labeled modeling assumption.

Let $F(t)$ be a coalition whose membership is fixed but whose profiles and magnitudes evolve over time, and write $Y(t) = v(F(t)) = D(t) \cdot R(t)$ with $D(t) = D(F(t))$ and $R(t) = R(F(t))$.

PROPOSITION 9.1—Growth Decomposition: *Suppose $R(t)$ is differentiable and positive. Since $D(t)$ is integer-valued, it is piecewise constant; let $t_1 < t_2 < \dots$ be its (isolated) transition times. Then:*

- (a) Between transitions, the growth rate of value is the growth rate of resources: $\dot{Y}/Y = \dot{R}/R$.
- (b) At a transition from D^- to D^+ , $\log Y$ jumps by $\log(D^+/D^-)$: rank gains contribute discrete growth episodes; rank losses contribute discrete contractions.
- (c) Over any interval $[0, T]$ containing transitions at $t_1 < \dots < t_K$,

$$\log \frac{Y(T)}{Y(0)} = \int_0^T \frac{\dot{R}(t)}{R(t)} dt + \sum_{k=1}^K \log \frac{D(t_k^+)}{D(t_k^-)}. \quad (9.1)$$

PROOF: (a)–(b) are immediate from $Y = D \cdot R$ with D locally constant; (c) integrates (a) between transitions and telescopes the jumps. Q.E.D.

If a smooth diversity dynamic is desired, replacing D by the entropic effective diversity $d_{\text{eff}} = e^S$ does not suffice: the entropic surrogate is non-monotone under coalition union, and the resulting value can report a strict loss from forming the grand coalition. A correct continuous economic layer requires constructing a smooth soft-rank functional on the spectrum of the capability matrix that is superadditive over all disjoint unions and recovers $v = D \cdot R$ exactly at the integer-rank anchors; under such a functional the growth decomposition becomes everywhere smooth and the telescoping sum is its integer limit. This construction is carried out in [Olfati-Saber \(2026p\)](#).

ASSUMPTION 9.2—Selection-Driven Capability Growth: *Magnitude growth is driven by adaptation: there exists $c > 0$ with $\dot{R}_0/R_0 \leq c \cdot \sigma_{\text{eff}}^2(F(t))$, where $R_0(t) = \sum_{i \in F(t)} r_i(t)$. This is a modeling assumption linking resource accumulation to the adaptive-improvement channel of the Fisher–Coalition Theorem (which, in its proved form, bounds expected fitness improvement, not resource growth).*

Under this assumption, homogenization suppresses growth through both channels of Proposition 9.1: $\sigma_{\text{eff}}^2 \rightarrow 0$ chokes the smooth component ($\dot{R}_0/R_0 \rightarrow 0$), and diversity suppression makes $D(t)$ non-increasing, eliminating positive jumps. For every choice of R built on R_0 —linear, variety, network, sector—the smooth channel inherits the bound, since each is a size- or activity-dependent multiple of R_0 with bounded multipliers along a fixed-membership sequence.

10. EXTRACTIVE INSTITUTIONS UNDER NONLINEAR RETURNS

The Structural Fragility Theorem of the foundational companion ([Olfati-Saber, 2026a](#), Theorem 6.4) establishes that in an extractive coalition $F = E \cup P$ with a small elite ($n_E \ll n_P$), population homogenization—the dynamic consequence of sustained extraction $\mathcal{E} > 1$, derived in the companion ([Olfati-Saber, 2026e](#))—triggers three simultaneous collapses: the effective diversity is asymptotically capped, $\limsup_t d_{\text{eff}}(F(t)) \leq 1 + n_E$; for $m > 1 + n_E$ the collapse threshold vanishes, $\xi_{\text{crit}}(F(t)) \rightarrow 0$, with near-blind spots in at least $m - 1 - n_E$ independent directions; and the population’s adaptive capacity is extinguished, $\sigma_{\text{eff}}^2(P(t)) \rightarrow 0$. The collapse chain is structural:

$$\mathcal{E} > 1 \Rightarrow \text{homogenization} \Rightarrow \left\{ d_{\text{eff}} \text{ capped, } \xi_{\text{crit}} \rightarrow 0, \sigma_{\text{eff}}^2(P) \rightarrow 0 \right\} \quad (m > 1 + n_E). \quad (10.1)$$

The framework makes no claim about the timing or probability of collapse, which would require a model of the shock process.

PROPOSITION 10.1—Larger Pie, Same Fragility: *The fragility conclusion is invariant to R : for any aggregate resource function—constant returns, increasing returns to variety, network effects, concave or sector-heterogeneous returns—the same homogenization hypotheses yield the full chain (10.1).*

PROOF: The effective diversity d_{eff} , collapse threshold $\xi_{\text{crit}} = \lambda_{\min}(C_F)$, profile variance σ_{eff}^2 , and density matrix ρ_F are structural quantities, invariant to R by the Spectral Invariance Theorem (Olfati-Saber, 2026a, Theorem 5.6). The hypotheses and proof of the fragility theorem reference only profiles and magnitudes, never R , so the conclusion holds verbatim under every specialization of this paper. *Q.E.D.*

REMARK 10.2: What R does change is the stake: under increasing returns, the value $v(F)$ destroyed when an exposed direction is struck is larger, and the extractable surplus sustaining $\mathcal{E} > 1$ is larger—making extraction more tempting precisely where its structural consequences are most costly. Combined with Section 9, extraction suppresses growth through both channels (smooth capability growth and rank jumps) regardless of R : extractive and inclusive societies diverge under every returns regime considered here. This gives the inclusive/extractive divergence of Acemoglu and Robinson (2012) a spectral mechanism: the divergence is a structural bifurcation at $\mathcal{E} = 1$ that holds for any R .

11. THE TWO-LAYER ARCHITECTURE: INHERITED BENEFITS

The five preceding cases developed domain-specific results—gains decompositions, Shapley comparisons, superadditivity, core stability—under five different specializations of R . In each case, the structural results of the foundational companion carried over without modification. Figure 1 summarizes the architecture; we now make the benefits of the inheritance precise.

11.1. What the Structural Layer Provides

Any choice of aggregate resource function R satisfying Definition 3.1 automatically inherits the following structural results, none of which requires re-derivation.

Diversity measurement. The profile density matrix ρ_F , von Neumann entropy $S(F)$, and effective diversity $d_{\text{eff}}(F) = e^{S(F)}$ are computed from profiles alone. These provide a complete characterization of the coalition’s diversity—whether the coalition is a team, a trade bloc, a platform, or a nation—without reference to R .

Vulnerability diagnosis. The null space of the capability matrix C_F identifies the directions in which the coalition has zero capability. The collapse threshold $\xi_{\text{crit}} = \lambda_{\min}(C_F)$ measures how close the coalition is to catastrophic failure along its weakest direction, and near homogeneity the threshold is priced linearly by the entropy. A coalition with $\xi_{\text{crit}} \rightarrow 0$ is one shock away from collapse, and this diagnosis holds whether returns are linear, concave, or network-enhanced.

Adaptive capacity. The Fisher–Coalition Theorem establishes that the expected adaptive improvement $\mathbb{E}_{\xi}[\Delta \bar{w}] = (\eta \kappa / m) \sigma_{\text{eff}}^2(F)$ vanishes if and only if $D(F) = 1$. Homogeneous coalitions have zero expected adaptive improvement regardless of R : a coalition that spans only one capability direction cannot adapt to shocks from any other direction.

The two-case marginal contribution structure. Proposition 3.3 establishes that every agent’s marginal contribution decomposes into two cases—span-expanding and span-redundant—based solely on the linear algebra of capability profiles. This dichotomy is inherited by every specialization of R , so the distinction between the diversity bonus and the resource bonus is general; and in generic position, the closed forms of Proposition 4.11 apply to every size scaling.

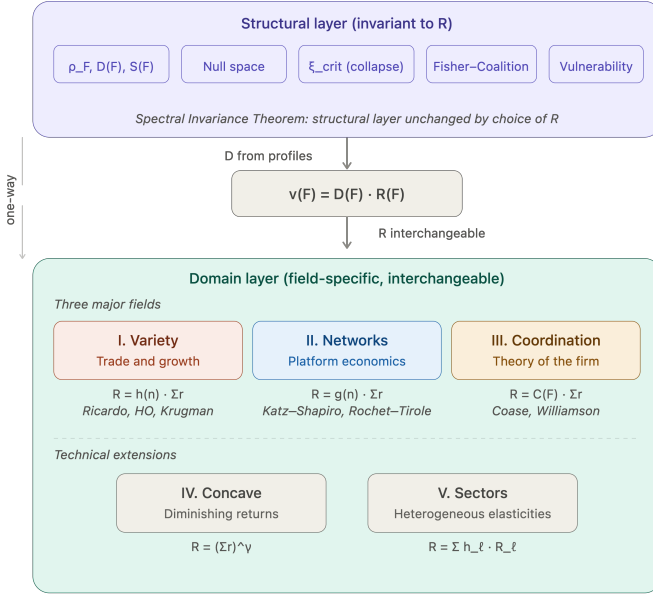


FIGURE 1.—The two-layer architecture of Coalition Intelligence Theory. The structural layer (top) is invariant to R by the Spectral Invariance Theorem: the profile density matrix, entropy, null space, collapse threshold, and adaptive capacity depend only on profiles and magnitudes. The value function $v = D \cdot R$ (center) connects the two layers, with D determined entirely by the structural layer and R interchangeable. The domain layer (bottom) is field-specific: three major fields—trade and growth (Case I, increasing returns to variety), platform economics (Case II, network-enhanced returns), and the theory of the firm (Case III, coordination-dependent returns)—are followed by two technical extensions (Cases IV–V, concave and sector-heterogeneous returns). Each specialization inherits the full structural theory without re-derivation. The one-way arrow indicates that the structural layer constrains the domain layer, but the domain layer does not feed back into the structural layer.

The extractive-institutions chain. Proposition 10.1 confirms that the fragility trajectory—extraction homogenizes, homogenization caps d_{eff} , opens near-blind spots, and freezes adaptation—holds under every returns regime. What R changes is the magnitude of the value destroyed and the size of the extractable surplus, but not the structural mechanism of collapse.

11.2. What the Domain Layer Determines

For each specialization of R , the domain layer determines the field-specific content: the gains decompositions (Theorems 4.5 and 8.2), the Shapley comparisons and closed forms (Theorem 4.9, Proposition 4.11), superadditivity conditions (Theorems 4.13 and 6.5), core stability and convexity (Theorems 4.15 and 4.16), the diversity threshold (Theorem 7.3), the curation trade-off (Proposition 5.7), and the coordination ceiling-and-floor architecture with the formation condition (Lemma 6.2, Theorem 6.4, Corollary 6.7). These results are specific to each R and must be derived case by case—which is the task the five preceding cases have accomplished.

The two-layer separation means that any future specialization of R —for a domain not considered here—inherits the structural layer in full and requires derivation only of its domain-specific results. The five cases developed here are illustrative, not exhaustive.

12. THE TRADITIONS RE-EXAMINED, AND PREDICTIONS

12.1. *Five Traditions*

Cooperative game theory: allocation without structure. Shapley (1953) introduced the Shapley value as the unique allocation satisfying efficiency, symmetry, null-player, and additivity. Bondareva (1963) and Shapley (1967) proved that a TU game has a non-empty core if and only if it is balanced. Shapley (1971) proved that convex games have non-empty cores containing the Shapley value, Ichiishi (1981) established the converse, Iñarra and Usategui (1993) relaxed convexity to average convexity, Sprumont (1990) obtained a related population-monotonicity result, and Maschler, Peleg and Shapley (1972) showed that in convex games all standard solution concepts select core allocations. These are complete, deep results—and none provides a structural account of the characteristic function itself; the decomposition $v = D \cdot R$ supplies that account. The present results locate the CIT game precisely within the classical hierarchy: strictly superadditive for all disjoint coalitions (Theorem 4.13) yet not convex for any variety premium on unrestricted configurations (Theorem 4.15), with convexity recovered exactly under full linear independence (Theorem 4.16). The product structure— D a matroid rank function, R the economic content—places the game at the intersection of matroid games and classical cooperative game theory; the core properties are determined by neither factor alone.

International trade. Ricardo (1817) and Ohlin (1933) established comparative advantage—technology-driven or endowment-driven—as the motive for trade; the linear case formalizes both as $D(F) \geq 2$. Krugman (1979) resolved the puzzle of intra-industry trade through increasing returns and product differentiation, with Dixit and Stiglitz (1977) supplying the preference foundation, and Helpman and Krugman (1985) synthesized the traditions. The decomposition $v = D \cdot h \cdot R$ makes the synthesis exact at the cooperative-game level: gains split additively into a classical and a variety component (Theorem 4.5), and which dominates is determined by rank alone (Corollary 4.6)—a sharp structural prediction where the synthesis left an empirical question.

Platform economics. Katz and Shapiro (1985) introduced network externalities and demand-side economies of scale; Rochet and Tirole (2003) extended the analysis to two-sided markets. These models treat users as homogeneous within each side. The decomposition $v = D \cdot g \cdot R$ adds within-side diversity as a value driver: equal-size, equal-resource networks are ordered by diversity (Lemma 5.3), the span-expansion premium grows with the network (Proposition 5.6), the winner-take-all dynamic is located in the convexity of total network value (Proposition 5.5), and the curation rule is scale-free (Proposition 5.7).

Theory of the firm. Coase (1937) asked why firms exist; the transaction cost tradition (Williamson, 1975, 1985) answered that market frictions make internal organization efficient. Each variant—transaction costs, property rights, team production, relational contracting—takes the market as the default, identifies a friction, and derives the firm as the remedy. The decomposition $v = D \cdot C \cdot R$ answers the prior question: the firm creates value because coalition formation under the coordination return is superadditive (Theorem 6.5), and the organized graph sustains the coordination that makes value positive (Theorem 6.4), while decentralized market topologies are bounded by the weakest-link ceiling (Lemma 6.2). The formation condition (6.9) requires no behavioral assumptions. Asset specificity, contract incompleteness, and opportunism are real empirical variables, but each maps to a definite object in the present framework—determinants of C , features of the allocation layer, or shadows of profile geometry—rather than causes of firm formation.

Growth and institutions. The growth decomposition connects to the endogenous-growth tradition: capability growth driven by Fisher–Coalition adaptive capacity parallels the innovation channel of Romer (1990), with diversity playing the role of the stock of ideas and rank

jumps providing the discrete innovation events. The extractive-institutions result gives the inclusive/extractive divergence of [Acemoglu and Robinson \(2012\)](#) a spectral mechanism that holds under every returns regime.

12.2. Predictions

The theory generates falsifiable predictions. Each is stated with its generating result and the observation that would disconfirm it.

Prediction 1 (rank-determined margin dominance). In measured trade decompositions, the inter-industry (classical) margin dominates the intra-industry (variety) margin for every bloc whose members span at least two independent capability directions, at every scale; the variety margin dominates only among profile-identical partners (Corollary 4.6). *Disconfirming observation:* the variety margin overtaking the classical margin in blocs with clearly distinct comparative advantages.

Prediction 2 (variety gains scale with partners). Gains from pure intra-industry trade among near-identical economies grow with the number of partners as $(h(n) - 1)R$, with $h(n) = n^{1/(\sigma-1)}$ calibrated by measured elasticities (Theorem 4.5, Proposition 4.3). *Disconfirming observation:* no partner-count effect on intra-industry gains at fixed total capacity.

Prediction 3 (persistent cross-agent rent under increasing returns). In large teams or blocs operating under increasing returns, the share of value attributable to cross-agent amplification stabilizes near $\alpha/(1 + \alpha)$ rather than vanishing, while under constant returns the corresponding share declines as $1/n$ (Corollary 4.12). *Disconfirming observation:* cross-agent rent shares vanishing with size in strongly increasing-returns settings, or failing to vanish in constant-returns settings.

Prediction 4 (curation toward diversity at comparable magnitudes). Platforms gain more from adding users whose capability profiles expand the network's span than from equally resourced redundant users, with the premium growing in network size as $g(n_S + 1)(R_S + r_i)$ (Propositions 5.6 and 5.7). *Disconfirming observation:* acquisition value independent of profile novelty at comparable user magnitudes, or a premium that shrinks as the network grows.

Prediction 5 (winner-take-all located in total value). Marginal user value increases with network size whenever total network value $n \cdot g(n)$ is convex, even under concave per-resource scaling; it decreases only under scalings that saturate (Proposition 5.5). *Disconfirming observation:* declining marginal user value in networks whose measured total-value curve is convex.

Prediction 6 (coordination as weakest link). Coordination capacity is bounded by the ratio of minimum to average degree (Lemma 6.2): adding a poorly connected member caps the coordination capacity of the entire coalition, regardless of density. *Disconfirming observation:* coalition coordination unaffected by the attachment structure of its least-connected member at fixed density.

Prediction 7 (merger value tracks integration). Post-merger value creation tracks the build-out of cross-connectivity, and mergers that combine under common ownership without building cross-ties destroy value, by the exact threshold (6.7) (Theorem 6.5). Internalization without integration forfeits the coordination benefit. *Disconfirming observation:* unintegrated mergers—few measured cross-ties—systematically creating value.

Prediction 8 (concavity threshold). Under diminishing returns with measured exponent γ_c , coalition formation among comparable agents is profitable only above the diversity threshold $D > n_F^{1-\gamma_c}$; homogeneous mergers destroy value (Theorem 7.3). *Disconfirming observation:* homogeneous large coalitions showing positive formation gains under verified concave returns.

13. CONCLUSION

Classical cooperative game theory takes the characteristic function as an unstructured exogenous primitive: it can divide any given pie, but it cannot prescribe how to build the coalition that produces the largest pie, because the structure of v is not specified. The decomposition $v(F) = D(F) \cdot R(F)$, constructed in the foundational companion and developed here under five nonlinear specializations, provides the structural foundations that classical theory left open.

The Spectral Invariance Theorem guarantees that the structural layer—diversity, vulnerability, adaptive capacity—is domain-independent; the domain layer is interchangeable by design. Two transformations follow. First, cooperative game theory moves from unstructured to transparent: the Shapley value, the core, and the gains from coalition formation all decompose into terms traceable to the factors of v —in generic position, into closed forms with an exact efficiency identity, for any size premium. Second, the theory moves from descriptive to prescriptive: the decomposition prescribes whom to add (span-expanding agents, at comparable magnitudes), when coalition formation is worthwhile (the diversity thresholds of Theorems 4.5 and 7.3), where the structural vulnerabilities lie, and whether the resulting allocation is stable. Fair allocation, stability, and organizational design are unified: the hiring problem, the trade problem, and the stability problem are the same problem viewed through different specializations of R .

The five cases establish new results in each domain:

Case	$R(F)$	Key results
I. Variety	$h(n_F) \cdot \sum r_i$	Gains decomposition (Th. 4.5); rank-determined margins (Cor. 4.6); strict gains for every agent and spanwise dominance (Th. 4.9); generic-position closed forms with exact efficiency (Prop. 4.11) and the persistent rent share $\alpha/(1 + \alpha)$ (Cor. 4.12); unconditional strict superadditivity (Th. 4.13); universal failure of convexity (Th. 4.15) and convexity under full independence with core membership (Th. 4.16)
II. Networks	$g(n_F) \cdot \sum r_i$	Diversity multiplier (Lem. 5.3); winner-take-all located in the convexity of $n \cdot g(n)$ (Prop. 5.5); growing span-expansion premium (Prop. 5.6); scale-free curation rule (Prop. 5.7)
III. Coordination	$\mathcal{C}(F) \cdot \sum r_i$	Weakest-link ceiling (Lem. 6.2); organizational floor (Th. 6.4); superadditivity under non-degradation with merger converse (Th. 6.5); value ratio equals capacity ratio (Cor. 6.6); formation condition (Cor. 6.7)
IV. Concave	$(\sum r_i)^{\gamma_c}$	Diversity threshold $D > n_F^{1-\gamma_c}$ (Th. 7.3)
V. Sectors	$\sum_\ell h_\ell R_\ell$	Sector-specific variety components (Th. 8.2)

Each case enters a major domain of economics, and in each the decomposition produces results the existing literature does not contain: the rank-determined margin dominance, the reversal from dissipating to persistent cross-agent rents, the universal non-convexity with its exact violation formula, the location of winner-take-all in total rather than per-resource value, the diversity threshold under concavity, and the micro-foundations of the firm from profile geometry and network structure without behavioral assumptions. The cases are illustrative, not exhaustive: any R satisfying Definition 3.1 inherits the full structural layer via Spectral Invariance.

The general characterization of core non-emptiness for the family $v = D \cdot h \cdot R$ —between the convex regime of full independence and the universally available non-convex configurations—remains the natural open problem; partial results on blocked inclusion under extraction appear

in the institutional companion (Olfati-Saber, 2026e). Beyond it, natural extensions include allowing R to depend jointly on size and spectrum (coupling the layers) and the dynamic theory whose need Section 9 makes explicit. The decomposition transforms cooperative game theory from a theory of division into a theory of design.

APPENDIX A: NOTATION

Symbol	Meaning
$\mathbf{q}_i = r_i \mathbf{s}_i$	capability vector; magnitude r_i , profile $\mathbf{s}_i \in \mathbb{S}^{m-1}$
$D(F), \rho_F, S(F), d_{\text{eff}}$	rank diversity; density matrix; entropy; effective diversity
$R(F), v(F)$	aggregate resource function; value $D \cdot R$
$h(n), g(n)$	variety premium; network scaling ($h(1) = g(1) = 1$)
$\chi(n), \chi_g(n), \chi_D(k)$	$nh(n); ng(n); \min(k, m)h(k)$
γ_c	concavity exponent, $R = (\sum r_i)^{\gamma_c}$
$h_\ell, R_\ell, n_{F,\ell}$	sector premium; sector resources; active count in sector ℓ
$\mathcal{C}(F), \lambda_2(L), \delta(\mathcal{G})$	coordination capacity; Fiedler eigenvalue; aggregate degree
$d_{\min}, d_{\max}, \bar{d}$	minimum, maximum, and average degree
$\iota(\mathcal{G}), (\theta, \psi)$	isoperimetric number; organizational parameters
$\phi_i^{\text{Sh}}, \gamma^h, \beta_i^h$	Shapley value; base coefficient; cross-agent rent (generic position)
$G(F)$	gains from coalition formation
$I_i(S)$	span-expansion indicator $\mathbf{1}[\mathbf{s}_i \notin \text{span}(S)]$
$C_F, \xi_{\text{crit}}, \sigma_{\text{eff}}^2$	capability matrix; collapse threshold; profile variance
$E, P, \mathcal{E}$	elites; masses; extraction ratio

REFERENCES

- Acemoglu, Daron, and James A. Robinson (2012), *Why Nations Fail*. Crown Business, New York. [23, 26]
- Alon, Noga (1985), “Eigenvalues and Expanders.” *Combinatorica*, 6 (2), 83–96. [18]
- Bondareva, Olga N. (1963), “Some Applications of Linear Programming Methods to the Theory of Cooperative Games.” *Problemy Kibernetiki*, 10, 119–139. [1, 25]
- Broda, Christian, and David E. Weinstein (2006), “Globalization and the Gains from Variety.” *Quarterly Journal of Economics*, 121 (2), 541–585. [7, 20, 21]
- Coase, Ronald H. (1937), “The Nature of the Firm.” *Economica*, 4 (16), 386–405. [25]
- Dixit, Avinash K., and Joseph E. Stiglitz (1977), “Monopolistic Competition and Optimum Product Diversity.” *American Economic Review*, 67 (3), 297–308. [25]
- Fiedler, Miroslav (1973), “Algebraic Connectivity of Graphs.” *Czechoslovak Mathematical Journal*, 23 (2), 298–305. []
- Heckscher, Eli (1919), “The Effect of Foreign Trade on the Distribution of Income.” *Ekonomisk Tidskrift*, 21, 497–512. []
- Helpman, Elhanan, and Paul R. Krugman (1985), *Market Structure and Foreign Trade*. MIT Press, Cambridge, MA. [7, 8, 25]
- Ichiishi, Tatsuro (1981), “Super-modularity: Applications to Convex Games and to the Greedy Algorithm for LP.” *Journal of Economic Theory*, 25 (2), 283–286. [1, 25]
- Iñarra, Elena, and José M. Usategui (1993), “The Shapley Value and Average Convex Games.” *International Journal of Game Theory*, 22 (1), 13–29. [25]
- Katz, Michael L., and Carl Shapiro (1985), “Network Externalities, Competition, and Compatibility.” *American Economic Review*, 75 (3), 424–440. [25]
- Krugman, Paul R. (1979), “Increasing Returns, Monopolistic Competition, and International Trade.” *Journal of International Economics*, 9 (4), 469–479. [2, 7, 8, 9, 13, 25]
- Maschler, Michael, Bezalel Peleg, and Lloyd S. Shapley (1972), “The Kernel and Bargaining Set for Convex Games.” *International Journal of Game Theory*, 1 (1), 73–93. [25]
- Mohar, Bojan (1989), “Isoperimetric Numbers of Graphs.” *Journal of Combinatorial Theory, Series B*, 47 (3), 274–291. [18]

- Ohlin, Bertil (1933), *Interregional and International Trade*. Harvard University Press, Cambridge, MA. [25]
- Olfati-Saber, Reza (2026a), “Collective Intelligence of Coalitions.” Working paper, Institute for Intelligence Science and Society. [2, 3, 4, 5, 10, 13, 17, 19, 22, 23]
- Olfati-Saber, Reza (2026e), “Collapse of Extractive Institutions in Non-Convex Coalition Games.” Working paper, Institute for Intelligence Science and Society. [22, 28]
- Olfati-Saber, Reza (2026p), “A Continuous Economic Layer for Coalition Value: The Soft Rank and the Forced Odd Weights.” Working paper, Institute for Intelligence Science and Society. [22]
- Ricardo, David (1817), *On the Principles of Political Economy and Taxation*. John Murray, London. [2, 13, 25]
- Rochet, Jean-Charles, and Jean Tirole (2003), “Platform Competition in Two-Sided Markets.” *Journal of the European Economic Association*, 1 (4), 990–1029. [25]
- Romer, Paul M. (1990), “Endogenous Technological Change.” *Journal of Political Economy*, 98 (5), S71–S102. [26]
- Shapley, Lloyd S. (1953), “A Value for n -Person Games.” In *Contributions to the Theory of Games II* (Harold W. Kuhn and Albert W. Tucker, eds.), 307–317, Princeton University Press, Princeton, NJ. [1, 25]
- Shapley, Lloyd S. (1967), “On Balanced Sets and Cores.” *Naval Research Logistics Quarterly*, 14 (4), 453–460. [1, 25]
- Shapley, Lloyd S. (1971), “Cores of Convex Games.” *International Journal of Game Theory*, 1 (1), 11–26. [1, 12, 13, 25]
- Sprumont, Yves (1990), “Population Monotonic Allocation Schemes for Cooperative Games with Transferable Utility.” *Games and Economic Behavior*, 2 (4), 378–394. [25]
- Williamson, Oliver E. (1975), *Markets and Hierarchies: Analysis and Antitrust Implications*. Free Press, New York. [25]
- Williamson, Oliver E. (1985), *The Economic Institutions of Capitalism*. Free Press, New York. [25]