

COLLECTIVE INTELLIGENCE OF COALITIONS

REZA OLFATI-SABER

Institute for Intelligence Science and Society

<https://institute.wisdomagent.com>

Classical cooperative game theory solves the allocation and stability problems for a characteristic function $v: 2^N \rightarrow \mathbb{R}$ supplied from outside. Because v is an unexplained primitive, the theory can allocate a surplus but cannot derive one, diagnose vulnerability, or measure adaptive capacity. This paper develops Coalition Intelligence Theory, a structural theory that equips v with decomposable causal factors. Agents carry capability vectors factored into a magnitude and a profile; the coalition's state is the profile density matrix ρ_F , and its value is $v(F) = D(F) \cdot R(F)$, rank diversity times an aggregate resource function. A Spectral Invariance Theorem separates a structural layer—density matrix, entropy, null space, collapse threshold, adaptive capacity—from an interchangeable domain layer. The spectrum of ρ_F organizes three independent roles of diversity: value creation, through superadditivity and an exact Shapley decomposition whose span-expansion bonus is a scarcity rent dissipating as $1/n$ once the coalition exceeds its capability dimension; resilience, through a collapse threshold that is positive iff profiles span the full space and is controlled linearly by the entropy near homogeneity; and adaptation, through a Fisher–Coalition theorem making expected improvement proportional to profile variance. A structural fragility theorem shows that extraction-driven homogenization destroys all three simultaneously—three independent collapses of one spectrum. Applications to extractive institutions and international trade, the latter unifying Ricardian and Heckscher–Ohlin comparative advantage, follow with seven falsifiable predictions.

KEYWORDS: coalition theory, collective intelligence, cooperative games, Shapley value, profile density matrix, von Neumann entropy, comparative advantage, extractive institutions.

August 15, 2026

1. INTRODUCTION

1.1. *The Problem*

Cooperative game theory, as formalized by von Neumann and Morgenstern (1944) and Shapley (1953), is a theory of division. Given a characteristic function $v: 2^N \rightarrow \mathbb{R}$ assigning a value to every coalition, the theory determines how to divide the value fairly (the Shapley value), whether the grand coalition is stable (the core and balancedness; Bondareva 1963, Shapley 1967), and when the fair allocation is also stable (convexity). These are deep results, and the seven decades since Shapley's contribution have produced a complete theory of allocation and stability for transferable-utility games.

But the theory takes v as a primitive—an unstructured exogenous table of 2^n numbers. It provides no account of where v comes from: what structural properties of agents generate a particular v , what makes some coalitions more valuable than others, or how to construct the coalition that produces the largest pie. The allocation problem—divide the given pie—is solved; the *design* problem—build the coalition that produces the largest pie—is not addressed, because v has no internal structure to exploit. Three problems follow. First, the construction problem: every economic application of the framework—the theory of the firm, international trade, institutional

analysis—has had to import the characteristic function from outside, from production functions, exchange technologies, or behavioral assumptions; no structural account of v grounded in the agents who compose the coalition has been provided within the theory itself. Second, the vulnerability problem: the characteristic function assigns a value but does not identify the directions in which that value can be destroyed, how close the coalition is to catastrophic failure, or whether it can adapt to unforeseen change—vulnerability, resilience, and adaptive capacity are central to institutional economics, organizational theory, and evolutionary biology, and have no formal home in cooperative games. Third, the allocation-under-structure problem: even given a structured value function, a decomposition of the Shapley value tracing each agent’s payoff to identifiable structural features—separating the return on own resources from the premium for expanding the coalition’s capabilities—is needed to connect allocation to design.

This is a striking state of affairs. The Shapley value tells you what each agent is worth but not *why*—not which structural feature of the agent generates the value. The core tells you whether the grand coalition is stable but not *what* makes it stable. The theory can divide a surplus; it cannot tell you where the surplus comes from, or whether it exists at all.

1.2. Conceptual Foundation

Rather than positing a value for each coalition, we ask what the agents are made of. An agent carries a capability vector $q_i \in \mathbb{R}_{++}^m$, factored into a magnitude $r_i = \|q_i\|$ —how much the agent can do—and a profile $s_i = q_i/r_i$ —what kind of thing the agent does. The coalition’s state is the profile density matrix $\rho_F = (1/n_F) \sum_{i \in F} s_i s_i^\top$, a positive semidefinite, unit-trace matrix whose spectrum encodes the coalition’s collective knowledge: its rank is the number of independent capability directions spanned, its von Neumann entropy exponentiates to the effective diversity, its null space records the directions of zero capability, and the smallest eigenvalue of the associated capability matrix is a collapse threshold. The value of a coalition is then not assumed but constructed: $v(F) = D(F) \cdot R(F)$, rank diversity times an aggregate resource function—and the factorization is a consequence of the capability-profile representation, not a modeling choice.

The construction resolves all three problems at once, and it does so with a two-level architecture. The structural layer—the density matrix and everything read off its spectrum—is invariant to the choice of R , by a Spectral Invariance Theorem; the domain layer, carried by R , is field-specific and interchangeable. One direction of dependence, never the reverse: any specialization of R inherits the complete structural theory without re-derivation. This paper is the first of a foundational pair: the companion (Olfati-Saber, 2026b) develops the decomposition of the value function and its nonlinear specifications—increasing returns, network effects, coordination-dependent value—while the dynamical and evolutionary layers, in which the spectrum acquires equations of motion and demography prices its maintenance, are developed in further companions (Olfati-Saber, 2026c,f), and the mechanism producing the homogenization studied in Section 6 is derived there and in the institutional companion (Olfati-Saber, 2026e).

1.3. Contributions

The framework yields eight analytically distinct capabilities. First, the explicit characteristic function itself—constructed from capabilities, not assumed. Second, a Superadditivity Theorem: agents with complementary profiles are worth strictly more together than apart, with the surplus computed exactly as cross terms $D(F)R_{F'} + D(F')R_F$, requiring no behavioral assumptions. Third, the two-layer architecture and Spectral Invariance. Fourth, the

vulnerability apparatus: the null space identifies blind spots exactly, and the collapse threshold $\xi_{\text{crit}} = \lambda_{\min}(C_F)$ is positive if and only if profiles span the full space—with a quantitative form: near homogeneity the threshold is controlled *linearly by the entropy*, $\xi_{\text{crit}} \leq r_{\max} n_F S(F) / ((m-1) \log 2)$, so the spectrum does not merely diagnose fragility but prices it. Fifth, a Fisher–Coalition Theorem: expected adaptive improvement under random shocks is proportional to profile variance, vanishing if and only if the coalition is homogeneous. Sixth, the triple role of diversity—value, resilience, adaptability—as three independent bottlenecks of one spectrum. Seventh, an exact Shapley decomposition $\phi_i^{\text{Sh}} = \gamma_i r_i + \beta_i$ into a rank-weighted base and a span-expansion bonus, together with closed forms in generic position: the base coefficient is the same for every agent— $(n+1)/2$ below the capability dimension and $m - m(m-1)/(2n)$ above it—and the bonus is $\beta_i = (R_F - r_i)K(K+1)/(2n(n-1))$ with $K = \min(m-1, n-1)$, so that the span-expansion premium is a *scarcity rent of position*, dissipating as $1/n$ once the coalition outgrows its capability dimension. Eighth, the transition from descriptive to prescriptive: whom to add, when formation is worthwhile, where the vulnerabilities lie.

Two applications close the theory. In extractive institutions, population homogenization simultaneously caps value, opens at least $m-1 - n_E$ blind-spot directions, and freezes adaptive capacity—with the value collapse carried by the effective diversity, the continuous spectral quantity that the integer-valued rank cannot track along the approach. In international trade, profile non-parallelism unifies Ricardian comparative advantage and Heckscher–Ohlin factor endowments as one geometric condition in two coordinate systems.

1.4. Organization

Section 2 introduces the primitives. Section 3 develops the coalition geometry. Section 4 builds the cooperative game, proves the Shapley decomposition, and derives the generic-position closed forms. Section 5 establishes the vulnerability apparatus, the Fisher–Coalition theorem, and Spectral Invariance. Section 6 applies the framework to extractive institutions and Section 7 to international trade. Section 8 re-examines the traditions the theory touches and states seven falsifiable predictions; Section 9 concludes. Quantitative claims are verified by exact enumeration and simulation in the Appendix.

2. PRIMITIVES

This section introduces the primitives: agents and capabilities, the density matrix and its spectral diversity measures, coordination and fidelity, collective intelligence, the value function and its allocation, and the vulnerability apparatus. All definitions are stated here; their consequences are developed in Sections 3–5.

2.1. Agents and Capabilities

DEFINITION 2.1—Agent and Capability Vector: An agent $i \in \mathcal{A} = \{1, \dots, n\}$ is characterized by a capability vector $q_i \in \mathbb{R}_{++}^m$, decomposed as $q_i = r_i \cdot s_i$, where $r_i = \|q_i\| > 0$ is the capability magnitude and $s_i = q_i/r_i$ is the capability profile, a point of the unit sphere $S^{m-1} \subset \mathbb{R}^m$. Since $q_i \in \mathbb{R}_{++}^m$, all profiles lie in the strictly positive orthant of S^{m-1} , so $s_i^\top s_j > 0$ for all i, j .

DEFINITION 2.2—Coalition: A coalition $F \subseteq \mathcal{A}$ has size $n_F = |F|$, capability matrix $Q_F = [q_1 | \dots | q_{n_F}] \in \mathbb{R}^{m \times n_F}$, profile matrix $S_F = [s_1 | \dots | s_{n_F}] \in \mathbb{R}^{m \times n_F}$, and rank diversity $D(F) = \text{rank}(S_F) \in \{1, \dots, \min(m, n_F)\}$ for $F \neq \emptyset$. We adopt the conventions $\text{span}(\emptyset) = \{0\}$ and $D(\emptyset) = 0$.

The magnitude–profile decomposition separates *how much* an agent can do from *what kind* of thing they do. Two agents with the same profile but different magnitudes are substitutes differing only in scale; two agents with different profiles are complements whose combination expands the coalition’s capability span. Rank diversity counts the independent capability directions the coalition spans—the dimension of the coalition’s knowledge.

2.2. Diversity

DEFINITION 2.3—Profile Density Matrix: The knowledge state of a non-empty coalition F is

$$\rho_F = \frac{1}{n_F} \sum_{i \in F} s_i s_i^\top \in \mathbb{R}^{m \times m}. \quad (2.1)$$

The matrix ρ_F is positive semidefinite with $\text{tr}(\rho_F) = 1$ and $\text{rank}(\rho_F) = \text{rank}(S_F) = D(F)$. Since each s_i has strictly positive entries, ρ_F has strictly positive entries. The density matrix is the central geometric object of the theory: its spectrum encodes every structural property of the coalition.

DEFINITION 2.4—Von Neumann Entropy: The diversity of coalition F is

$$S(F) = -\text{tr}(\rho_F \log \rho_F) = -\sum_{k=1}^m \lambda_k \log \lambda_k, \quad (2.2)$$

where $\{\lambda_k\}$ are the eigenvalues of ρ_F and $0 \log 0 = 0$. Entropy is measured in nats.

DEFINITION 2.5—Effective Diversity: $d_{\text{eff}}(F) = e^{S(F)} \in [1, \min(n_F, m)]$.

Effective diversity is the exponential of the von Neumann entropy—a continuous measure interpolating between the discrete rank diversity $D(F)$ and the maximal diversity $\min(n_F, m)$. When all nonzero eigenvalues of ρ_F are equal, effective diversity equals rank diversity; when they are unequal, effective diversity is strictly less than rank, reflecting the concentration of capability in a few directions. The distinction is not cosmetic: rank is integer-valued and can remain maximal while the coalition concentrates arbitrarily close to a single direction, whereas effective diversity tracks the concentration continuously—a fact that carries the asymptotic statements of Section 6.

2.3. Coordination and Fidelity

DEFINITION 2.6—Interaction Network: The coalition’s interaction structure is a weighted graph $G = (F, E, W)$ with adjacency matrix $W = [w_{ij}]$, $w_{ij} \geq 0$, and aggregate degree $\delta(G) = \sum_i \sum_j w_{ij}$. We assume $\delta(G) > 0$ whenever $n_F \geq 2$.

DEFINITION 2.7—Graph Laplacian: The graph Laplacian is $L = D_W - W$ where $D_W = \text{diag}(\sum_j w_{ij})$ is the degree matrix. The algebraic connectivity $\lambda_2(L) \geq 0$ is the second-smallest eigenvalue, with $\lambda_2(L) > 0$ iff G is connected (Fiedler, 1973).

DEFINITION 2.8—Coordination Capacity: For $n_F \geq 2$, $C(F) = \lambda_2(L)/\lambda_2^{\max} \in [0, 1]$, where $\lambda_2^{\max} = \delta(G)/(n_F - 1)$. For $n_F = 1$, $C(\{i\}) = 1$ by convention.

DEFINITION 2.9—Signal Fidelity: $\mathcal{F}(F) = 1/(1 + A) \in (0, 1]$, where $A \geq 0$ is total information distortion. For $n_F = 1$, $\mathcal{F}(\{i\}) = 1$.

2.4. Collective Intelligence

DEFINITION 2.10—Collective Intelligence: The collective intelligence of coalition F is

$$\text{CI}(F) = e^{S(F)} \cdot C(F) \cdot \mathcal{F}(F) \in [0, \min(n_F, m)]. \quad (2.3)$$

Collective intelligence is effective diversity times coordination capacity times signal fidelity—three measurable factors, each a bottleneck. The absence of any one renders the coalition unable to accomplish an objective or create value, and the multiplicative structure encodes precisely this: $S(F) = 0$ forces $e^{S(F)} = 1$ and hence $\text{CI} \leq C(F) \cdot \mathcal{F}(F) \leq 1$; $C(F) = 0$ forces $\text{CI} = 0$; $\mathcal{F}(F) \rightarrow 0$ forces $\text{CI} \rightarrow 0$. For a single agent a , $\rho_a = s_a s_a^\top$ has rank 1, so $S(\{a\}) = 0$, and with the conventions of Definitions 2.8–2.9, $\text{CI}(\{a\}) = 1$. This is the first formal characterization of collective intelligence as a mathematical quantity derived from the structural properties of the group.

2.5. Value and Allocation

DEFINITION 2.11—Generalized Coalition Value Function: The coalition value function is

$$v(F) = D(F) \cdot R(F), \quad (2.4)$$

where $R : 2^{\mathcal{A}} \rightarrow \mathbb{R}_+$ is an aggregate resource function with $R(\emptyset) = 0$ and $R(F) > 0$ for $F \neq \emptyset$. Together with $D(\emptyset) = 0$, this gives $v(\emptyset) = 0$. Throughout this paper, we work with the canonical linear case $R(F) = \sum_{i \in F} r_i$, yielding

$$v(F) = D(F) \cdot \sum_{i \in F} r_i. \quad (2.5)$$

The value function is not assumed; it is constructed from the agents' capabilities. Rank diversity enters as the structural factor—the dimension of the coalition's knowledge—and the aggregate resource function encodes the domain-specific content. Under the linear specification, all gains from coalition formation arise from profile diversity: the coalition's value equals the number of independent capability directions it spans, times the total resource base.

DEFINITION 2.12—Shapley Value: Agent i 's Shapley value is

$$\phi_i^{\text{Sh}}(F) = \sum_{S \subseteq F \setminus \{i\}} \frac{|S|! (|F| - |S| - 1)!}{|F|!} [v(S \cup \{i\}) - v(S)]. \quad (2.6)$$

2.6. Institutional Structure

DEFINITION 2.13—Extractive Institution: A coalition $F = E \cup P$ consisting of elites E and masses P with $|E| = n_E \ll n_P = |P|$ operates under an actual allocation $(\phi_E^{\text{actual}}, \phi_P^{\text{actual}})$ that is budget-balanced: $\phi_E^{\text{actual}} + \phi_P^{\text{actual}} = v(F)$. The coalition is extractive if $\mathcal{E} = \phi_E^{\text{actual}} / \phi_E^{\text{Sh}} > 1$: the elites earn more than their fair Shapley share.

2.7. Vulnerability

DEFINITION 2.14—Null Space and Directional Capability: The null space of F is $\text{null}(F) = \ker(\rho_F)$, with $\dim(\text{null}(F)) = m - D(F)$. The capability matrix is $C_F = \sum_{i \in F} r_i s_i s_i^\top$ and the directional capability is $c_F(u) = u^\top C_F u$ for $u \in S^{m-1}$. Since all $r_i > 0$, $\ker(C_F) = \ker(\rho_F) = \text{null}(F)$.

DEFINITION 2.15—Collapse Threshold: The collapse threshold is

$$\xi_{\text{crit}}(F) = \min_{u \in S^{m-1}} c_F(u) = \lambda_{\min}(C_F).$$

The collapse threshold measures the coalition's capability along its weakest direction—the direction most vulnerable to an environmental shock. A coalition with $\xi_{\text{crit}} > 0$ can withstand shocks from any direction; a coalition with $\xi_{\text{crit}} = 0$ has at least one blind spot—a direction in which it has zero capability and is therefore catastrophically exposed.

3. COALITION GEOMETRY

This section develops the geometric foundations: how similarity between agents affects diversity, how adding a member changes the coalition's span, and what bounds entropy imposes on the knowledge state. These results depend only on profiles and are structural in the sense that they hold for any economic specification.

THEOREM 3.1—Complementarity: *Let $i, j \in \mathcal{A}$ with similarity $\sigma = s_i^\top s_j \in (0, 1]$. The coalition $F = \{i, j\}$ satisfies: (a) if $\sigma = 1$, then $D(F) = 1$, $S(F) = 0$, $d_{\text{eff}} = 1$; (b) as $\sigma \rightarrow 0^+$, $D(F) = 2$, $S(F) \rightarrow \log 2$, $d_{\text{eff}} \rightarrow 2$; (c) $d_{\text{eff}}(F)$ is strictly decreasing in σ over $(0, 1]$, with the exact spectrum $\lambda_{\pm} = (1 \pm \sigma)/2$.*

PROOF: The density matrix is $\rho_F = \frac{1}{2}(s_i s_i^\top + s_j s_j^\top)$. All eigenvalues outside $\text{span}(s_i, s_j)$ are zero. Applying ρ_F to $s_i + s_j$ gives $\rho_F(s_i + s_j) = \frac{1+\sigma}{2}(s_i + s_j)$, an eigenvector with eigenvalue $\lambda_+ = (1 + \sigma)/2$; similarly $s_i - s_j$ is an eigenvector with $\lambda_- = (1 - \sigma)/2$. Since $\sigma > 0$, $\lambda_+ > \lambda_- \geq 0$ and $\lambda_+ + \lambda_- = 1$. (a) At $\sigma = 1$, $\lambda_- = 0$, so $D = 1$, $S = 0$, $d_{\text{eff}} = 1$. (b) As $\sigma \rightarrow 0^+$, $\lambda_{\pm} \rightarrow 1/2$, so $S \rightarrow \log 2$ and $d_{\text{eff}} \rightarrow 2$. (c) $S(\sigma) = -\frac{1+\sigma}{2} \log \frac{1+\sigma}{2} - \frac{1-\sigma}{2} \log \frac{1-\sigma}{2}$ has $dS/d\sigma = -\frac{1}{2} \log \frac{1+\sigma}{1-\sigma} < 0$ on $(0, 1)$, so $d_{\text{eff}} = e^{S(\sigma)}$ is strictly decreasing. *Q.E.D.*

REMARK 3.2: The limit in part (b) is approached but not attained: the positive-orthant constraint forces $\sigma > 0$, so a two-agent coalition can come arbitrarily close to, but never reach, $d_{\text{eff}} = 2$.

The Complementarity Theorem establishes that diversity is a continuous, monotone function of profile dissimilarity: identical agents contribute nothing to diversity, maximally dissimilar agents contribute maximally. The next result governs how diversity grows as the coalition expands.

THEOREM 3.3—Coverage: *For $\emptyset \neq F \subseteq F'$, $D(F) \leq D(F')$. Moreover, for any F (possibly empty) and any agent j ,*

$$D(F \cup \{j\}) = \begin{cases} D(F) + 1 & \text{if } s_j \notin \text{span}(F), \\ D(F) & \text{if } s_j \in \text{span}(F). \end{cases} \quad (3.1)$$

PROOF: The inclusion $\text{span}(F) \subseteq \text{span}(F')$ gives $D(F) \leq D(F')$. The dichotomy follows from the dimension formula: $\dim(\text{span}(F) + \text{span}(s_j)) = D(F) + 1 - \dim(\text{span}(F) \cap \text{span}(s_j))$, where the intersection is $\{0\}$ or $\text{span}(s_j)$. For $F = \emptyset$ the conventions give $D(\{j\}) = 1 = D(\emptyset) + 1$. *Q.E.D.*

The Coverage Theorem is the mechanism through which diversity enters the value function: adding a member whose profile lies outside the current span raises rank diversity by exactly one; adding a spanned member leaves it unchanged. This dichotomy—span-expanding versus span-redundant—drives the Shapley decomposition below.

THEOREM 3.4—Entropy Bounds: $0 \leq S(F) \leq \log \min(n_F, m)$. *The lower bound holds with equality iff $D(F) = 1$. The upper bound holds with equality iff $\text{rank}(\rho_F) = \min(n_F, m)$ and all nonzero eigenvalues of ρ_F are equal.*

PROOF: Let $k = \text{rank}(\rho_F) \leq \min(n_F, m)$ and $\lambda_1, \dots, \lambda_k > 0$ the nonzero eigenvalues, summing to one. The lower bound: each $-\lambda_\ell \log \lambda_\ell \geq 0$, with equality for $\lambda_\ell > 0$ only at $\lambda_\ell = 1$; since eigenvalues sum to one, there is then a single nonzero eigenvalue, i.e. $D(F) = 1$. The upper bound: $f(x) = -x \log x$ is strictly concave, so Jensen’s inequality over the k nonzero eigenvalues gives $S(F) \leq \log k \leq \log \min(n_F, m)$, with equality iff all nonzero eigenvalues equal $1/k$ and $k = \min(n_F, m)$. *Q.E.D.*

For $n_F \geq 2$ and $m \geq 2$, the upper bound is a strict supremum within the model: ρ_F has strictly positive entries, so by Perron–Frobenius its largest eigenvalue is simple, which precludes the equal nonzero eigenvalues that equality would require.

4. COOPERATIVE GAMES

We now turn to value creation and allocation under the diversity-weighted value function. The results of this section are economic: they depend on the choice of R (here the linear specification), unlike the structural results of Sections 3 and 5.

THEOREM 4.1—Shapley (1953): *The Shapley value is the unique allocation satisfying efficiency, symmetry, the null player property, and additivity.*

THEOREM 4.2—Marginal Contribution: *Under $v(F) = D(F) \cdot \sum_j r_j$, for any $S \subseteq \mathcal{A} \setminus \{i\}$ (including $S = \emptyset$),*

$$v(S \cup \{i\}) - v(S) = \mathbf{1}[s_i \notin \text{span}(S)] \cdot R_S + D(S \cup \{i\}) \cdot r_i, \quad (4.1)$$

where $R_S = \sum_{j \in S} r_j$ and $R_\emptyset = 0$.

PROOF: Expanding, $v(S \cup \{i\}) - v(S) = [D(S \cup \{i\}) - D(S)]R_S + D(S \cup \{i\})r_i$; by Theorem 3.3, $D(S \cup \{i\}) - D(S) = \mathbf{1}[s_i \notin \text{span}(S)]$. *Q.E.D.*

The marginal contribution has two terms: a *span-expansion bonus* $\mathbf{1}[s_i \notin \text{span}(S)] \cdot R_S$, earned only when the agent’s profile expands the coalition’s span, and a *resource contribution* $D(S \cup \{i\}) \cdot r_i$ —own magnitude amplified by the diversity of the resulting coalition. The span-expansion bonus is the mechanism through which profile complementarity creates value: an agent who brings a new capability direction unlocks additional value from every existing member’s resources.

THEOREM 4.3—Superadditivity: *For disjoint non-empty F, F' with $\text{span}(F) \cap \text{span}(F') = \{0\}$,*

$$v(F \cup F') = v(F) + v(F') + D(F) \cdot R_{F'} + D(F') \cdot R_F > v(F) + v(F'). \quad (4.2)$$

PROOF: By the Grassmann dimension formula, $D(F \cup F') = D(F) + D(F')$. Expanding $v(F \cup F') = (D(F) + D(F'))(R_F + R_{F'})$ and subtracting yields the cross terms, which are strictly positive. *Q.E.D.*

REMARK 4.4: Without span-disjointness the value function remains weakly superadditive: $D(F \cup F') \geq \max(D(F), D(F'))$ gives $v(F \cup F') \geq v(F) + v(F')$.

The Superadditivity Theorem is the micro-foundation of coalition formation. Agents with complementary capabilities are worth strictly more together than apart—a theorem requiring no behavioral assumptions, no market friction, and no contracting technology. The cross terms are the explicit surplus from complementarity: each coalition’s diversity multiplies the other’s resources.

4.1. Shapley Decomposition

THEOREM 4.5—Shapley Decomposition: *Under $v(F) = D(F) \cdot \sum_{j \in F} r_j$, the Shapley value of agent $i \in F$ admits the exact decomposition*

$$\phi_i^{\text{Sh}}(F) = \gamma_i(F) \cdot r_i + \beta_i(F), \quad (4.3)$$

where

$$\gamma_i(F) = \sum_{S \subseteq F \setminus \{i\}} \frac{|S|!(n_F - |S| - 1)!}{n_F!} \cdot D(S \cup \{i\}) = \mathbb{E}_\pi[D(S_\pi \cup \{i\})] \quad (4.4)$$

is the expected rank of the predecessor-plus-self set under a uniformly random ordering π of F , and

$$\beta_i(F) = \sum_{S \subseteq F \setminus \{i\}} \frac{|S|!(n_F - |S| - 1)!}{n_F!} \cdot \mathbf{1}[s_i \notin \text{span}(S)] \cdot R_S \quad (4.5)$$

is the span-expansion bonus. Moreover, $1 \leq \gamma_i(F) \leq D(F)$ and $\beta_i(F) \geq 0$, with $\gamma_i(F) = 1$ iff $D(F) = 1$, and $\beta_i(F) = 0$ iff $s_i \in \text{span}(S)$ for every non-empty $S \subseteq F \setminus \{i\}$.

PROOF: Substituting the marginal contribution (4.1) into the Shapley formula (2.6) and separating terms yields the decomposition with γ_i and β_i as defined. For the bounds on γ_i : the Coverage Theorem gives $1 = D(\{i\}) \leq D(S \cup \{i\}) \leq D(F)$ for every S , and the Shapley weights are non-negative and sum to one. For the lower characterization: $\gamma_i = 1$ requires $D(S \cup \{i\}) = 1$ for all S , which holds iff all profiles are parallel, iff $D(F) = 1$. For β_i : each summand is a product of a non-negative weight, an indicator, and $R_S \geq 0$; the vanishing characterization follows because the indicator must vanish on every S with $R_S > 0$, i.e. every non-empty S . *Q.E.D.*

COROLLARY 4.6—Positivity of Shapley Values: *Under $v(F) = D(F) \cdot \sum_j r_j$, every agent’s Shapley value satisfies $\phi_i^{\text{Sh}}(F) \geq r_i > 0$.*

PROOF: $\phi_i^{\text{Sh}} = \gamma_i r_i + \beta_i$ with $\gamma_i \geq 1$ and $\beta_i \geq 0$. *Q.E.D.*

The decomposition separates the two channels through which coalition structure affects an individual’s payoff: the rank-weighted base $\gamma_i r_i$ is the return on own resources amplified by

the coalition's diversity, and the span-expansion bonus β_i is the additive premium for contributing a new capability direction. Positivity ensures that no agent can be exploited under the fair allocation. The decomposition also makes the allocation prescriptive: it identifies which structural feature of each agent—magnitude or profile—generates their value, and it traces the surplus to the span-expansion mechanism of the Coverage Theorem. In one important regime the decomposition admits closed forms.

PROPOSITION 4.7—Generic-Position Shapley Values: *Say the profiles of F are in generic position if every subset of at most m profiles is linearly independent. Then, writing $n = n_F$, $R_F = \sum_j r_j$, and $K = \min(m - 1, n - 1)$:*

(a) *the base coefficient is the same for every agent,*

$$\gamma_i(F) = \begin{cases} \frac{n+1}{2}, & n \leq m, \\ m - \frac{m(m-1)}{2n}, & n \geq m; \end{cases} \quad (4.6)$$

(b) *the span-expansion bonus is*

$$\beta_i(F) = (R_F - r_i) \frac{K(K+1)}{2n(n-1)}; \quad (4.7)$$

(c) *the two forms satisfy the efficiency identity $\sum_i \phi_i^{\text{Sh}} = v(F)$ exactly, in both regimes.*

PROOF: See Appendix B.

Q.E.D.

COROLLARY 4.8—Dissipation of Span Rents: *In generic position with $n \geq m$: the aggregate span-expansion bonus is $\sum_i \beta_i = \frac{m(m-1)}{2n} R_F$, so the bonus share of total value is*

$$\frac{\sum_i \beta_i}{v(F)} = \frac{m-1}{2n} \rightarrow 0, \quad (4.8)$$

and every individual bonus is $O(R_F m^2/n^2)$ relative to a per-capita base of order mR_F/n . As the coalition grows past its capability dimension, the fair allocation converges to proportionality in magnitudes with the common coefficient $\gamma \rightarrow m$.

PROOF: Sum (4.7) over i with $K = m - 1$ and divide by $v = mR_F$; the individual rates are read off (4.6)–(4.7). *Q.E.D.*

The economics of the closed forms deserves emphasis. In generic position all agents receive the *same* structural amplification: heterogeneity of fair payoffs comes from magnitudes, not from position in profile space. And the span-expansion premium is revealed as a *scarcity rent of position*: it is large exactly while independent directions are scarce relative to members ($n \leq m$, where the bonus grows with $K = n - 1$), and it dissipates as $1/n$ once the coalition outgrows its capability dimension—in a large generic coalition, almost every arrival finds their direction already spanned by some predecessor set, and the rent for novelty accrues only in the early, low-rank segments of the random order. The prescription is sharp: distinctive profiles command premia in small or dimension-starved coalitions, and command only their magnitude in large saturated ones.

5. VULNERABILITY AND RESILIENCE

The preceding sections established that diversity drives value creation and determines fair allocation. This section shows that diversity plays two additional and independent roles: it determines resilience—the coalition’s ability to withstand shocks—and it governs adaptive capacity—the coalition’s ability to improve its fitness in response to environmental change. Together, value, resilience, and adaptability are three mathematically distinct roles of diversity, each derived from the spectrum of the same density matrix.

THEOREM 5.1—Null Space: *If $u \in \text{null}(F)$, then $c_F(u) = 0$. The coalition has exactly $m - D(F)$ independent directions of zero capability.*

PROOF: For $u \in \ker(\rho_F)$, $u^\top \rho_F u = (1/n_F) \sum_{i \in F} (s_i^\top u)^2 = 0$. Each squared term is non-negative, so $s_i^\top u = 0$ for all $i \in F$. Therefore $c_F(u) = \sum_i r_i (s_i^\top u)^2 = 0$. The dimension count follows from rank-nullity. *Q.E.D.*

LEMMA 5.2—Correlated Exposure: *When $D(F) = 1$, so that $s_i = s^*$ for all $i \in F$, all agents have identical exposure $e_i = s^{*\top} \xi$ to any random shock ξ , and $\text{Corr}(e_i, e_j) = 1$ for all $i, j \in F$.*

PROOF: Since $s_i = s^*$ for all i , the exposures coincide as random variables. *Q.E.D.*

The Null Space Theorem makes the connection between diversity and vulnerability exact: every direction missing from the coalition’s capability span is a direction of complete exposure. A coalition with $D(F) < m$ has $m - D(F)$ independent blind spots, and the Correlated Exposure Lemma shows that a homogeneous coalition ($D = 1$) is the extreme case: all agents are exposed in exactly the same way, so the coalition has no capacity to hedge.

THEOREM 5.3—Collapse Threshold: *The collapse threshold satisfies the following properties:*

- (a) $\xi_{\text{crit}}(F) = 0$ iff $D(F) < m$.
- (b) $\xi_{\text{crit}}(F) > 0$ iff $D(F) = m$.
- (c) *Entropy control.* For $m \geq 2$, if $S(F) \leq \log 2$, then

$$\lambda_{\min}(\rho_F) \leq \frac{S(F)}{(m-1)\log 2}, \quad \xi_{\text{crit}}(F) \leq \frac{r_{\max} n_F S(F)}{(m-1)\log 2} = \frac{r_{\max} n_F \log d_{\text{eff}}(F)}{(m-1)\log 2}, \quad (5.1)$$

where $r_{\max} = \max_{i \in F} r_i$. In particular, along any sequence of fixed-size coalitions with uniformly bounded magnitudes, $S(F(t)) \rightarrow 0$ forces $\xi_{\text{crit}}(F(t)) \rightarrow 0$.

- (d) For equal magnitudes ($r_i = r$), $\xi_{\text{crit}}(F) = n_F r \cdot \lambda_{\min}(\rho_F) \leq n_F r / m$, with strict inequality for $n_F \geq 2$.

PROOF: (a)–(b). If $D(F) < m$, Theorem 5.1 gives $c_F(u) = 0$ for $u \in \text{null}(F) \cap S^{m-1}$, so $\xi_{\text{crit}} = 0$. If $D(F) = m$, then ρ_F is positive definite and $c_F(u) > 0$ for all $u \in S^{m-1}$; compactness and continuity yield $\xi_{\text{crit}} > 0$.

(c). Write the eigenvalues of ρ_F in non-increasing order $\lambda_1 \geq \dots \geq \lambda_m \geq 0$, summing to one. First, $\lambda_1 \geq 1/2$: otherwise every eigenvalue is below $1/2$, and $S(F) = \sum_k \lambda_k \log(1/\lambda_k) > \log 2 \sum_k \lambda_k = \log 2$, contradicting the hypothesis. Second, by the grouping (chain-rule) property of Shannon entropy applied to the partition $\{\lambda_1\} \cup \{\lambda_2, \dots, \lambda_m\}$, $S(F) \geq h(\lambda_1)$, where $h(x) = -x \log x - (1-x) \log(1-x)$ is the binary entropy. For $x \geq 1/2$ we have $1-x \leq$

$1/2$, so $-\log(1-x) \geq \log 2$ and $h(x) \geq -(1-x)\log(1-x) \geq (1-x)\log 2$. Combining, $1 - \lambda_1 \leq h(\lambda_1)/\log 2 \leq S(F)/\log 2$. Third, the $m-1$ eigenvalues below the top sum to $1 - \lambda_1$, so $\lambda_{\min}(\rho_F) = \lambda_m \leq (1 - \lambda_1)/(m-1) \leq S(F)/((m-1)\log 2)$. Finally, the Loewner comparison $C_F = \sum_i r_i s_i s_i^\top \preceq r_{\max} \sum_i s_i s_i^\top = r_{\max} n_F \rho_F$ gives $\xi_{\text{crit}}(F) = \lambda_{\min}(C_F) \leq r_{\max} n_F \lambda_{\min}(\rho_F)$, and $S = \log d_{\text{eff}}$ by Definition 2.5. The qualitative statement follows since $S(F(t)) \leq \log 2$ eventually.

(d). When $r_i = r$, $C_F = r n_F \rho_F$, so $\xi_{\text{crit}} = n_F r \cdot \lambda_{\min}(\rho_F)$. Since eigenvalues sum to one, $\lambda_{\min}(\rho_F) \leq 1/m$. Strictness for $n_F \geq 2$: ρ_F has strictly positive entries, so by Perron–Frobenius its largest eigenvalue is simple; if $\lambda_{\min} = 1/m$ then all eigenvalues would equal $1/m$, contradicting simplicity for $m \geq 2$. Q.E.D.

LEMMA 5.4—Diversity–Resilience: *For a coalition F , directional capability is positive, $c_F(u) > 0$ for all shock directions $u \in S^{m-1}$, iff $D(F) = m$.*

PROOF: Combine Theorem 5.1 and Theorem 5.3(b). Q.E.D.

Part (c) of the Collapse Threshold Theorem is the key link between diversity and fragility, and it is quantitative: near homogeneity the collapse threshold is controlled *linearly* by the entropy, with an explicit constant. As a coalition homogenizes—as its entropy drops toward zero—the threshold vanishes at least as fast as the entropy itself, regardless of how large or well-resourced the coalition is. A coalition can command abundant resources and arbitrarily high total capability while being one shock away from catastrophic failure, and the spectrum prices the distance to that failure.

THEOREM 5.5—Fisher–Coalition (See Table I) : *Under $\xi \sim \text{Unif}(S^{m-1})$ and fitness $w_i = s_i^\top \xi$, with selection intensity $\eta > 0$ and heritability $\kappa > 0$,*

$$\mathbb{E}_\xi[\Delta \bar{w}] = \frac{\eta \kappa}{m} \cdot \sigma_{\text{eff}}^2(F), \quad (5.2)$$

where $\sigma_{\text{eff}}^2(F) = (1/n_F) \sum_{i \in F} \|s_i - \bar{s}\|^2$. In particular, $\sigma_{\text{eff}}^2(F) = 0$ iff $D(F) = 1$.

PROOF: The mean fitness of the coalition under shock ξ is $\bar{w}(\xi) = \bar{s}^\top \xi$ with $\bar{s} = (1/n_F) \sum_i s_i$. The variance of fitness across members is

$$\text{Var}_F(w_i) = \frac{1}{n_F} \sum_{i \in F} ((s_i - \bar{s})^\top \xi)^2 = \xi^\top \Sigma_s \xi, \quad \Sigma_s = \frac{1}{n_F} \sum_{i \in F} (s_i - \bar{s})(s_i - \bar{s})^\top.$$

By Fisher’s fundamental theorem, the change in mean fitness under selection is proportional to the variance in fitness: $\Delta \bar{w}(\xi) = \eta \kappa \cdot \xi^\top \Sigma_s \xi$ for a fixed shock direction. Averaging over $\xi \sim \text{Unif}(S^{m-1})$ and using $\mathbb{E}[\xi \xi^\top] = (1/m)I$,

$$\mathbb{E}_\xi[\Delta \bar{w}] = \eta \kappa \cdot \text{tr}(\Sigma_s \mathbb{E}[\xi \xi^\top]) = \frac{\eta \kappa}{m} \text{tr}(\Sigma_s) = \frac{\eta \kappa}{m} \sigma_{\text{eff}}^2(F).$$

Finally, $\sigma_{\text{eff}}^2(F) = 0$ iff $s_i = \bar{s}$ for all i , iff $D(F) = 1$. Q.E.D.

The Fisher–Coalition Theorem completes the triple role of diversity. Value creation depends on rank diversity D ; resilience depends on whether $D = m$ and on the smallest eigenvalue of the capability matrix; adaptive capacity depends on the effective profile variance, which vanishes if and only if the coalition is homogeneous. A homogeneous coalition ($D = 1$) fails

TABLE I
ECONOMIC SYSTEMS AS INSTANCES OF THE FISHER–COALITION THEOREM

System	Agents	Profile s_i	Shock ξ	Selection η	Heritability κ
Diversified firm	Business divisions	Product-line capability mix	Demand shifts across segments	Budget reallocation to performing units	Divisions retain competency
Financial portfolio	Assets / funds	Factor exposure vector	Market regime change	Rebalancing toward outperformers	Asset retains its factor loading
National economy	Export sectors	Sectoral composition	Terms-of-trade / commodity swing	Capital and labor flow to favored sectors	Sectors retain productive identity
Supply chain	Suppliers	Geographic / logistic exposure	Disruption (port closure, tariff)	Order volume shifts to unaffected sources	Supplier retains routing capability
Monetary union	Member economies	Sectoral structure of GDP	Asymmetric demand shock	Fiscal transfers and labor mobility	Receiving economy retains structure

on all three counts: its value is $\sum_i r_i$ with no surplus, its collapse threshold is zero in every direction orthogonal to the common profile, and its expected adaptive improvement is zero. The three collapses are independent—a coalition can have high diversity and low resilience (spanning many directions unevenly), or high resilience and low adaptive capacity (diverse but clustered profiles)—and they are unified by the density matrix spectrum.

5.1. Spectral Invariance

THEOREM 5.6—Spectral Invariance: *The following quantities depend only on profiles $\{s_i\}_{i \in F}$ and magnitudes $\{r_i\}_{i \in F}$, and are invariant to the choice of R : (a) ρ_F and $S(F)$; (b) $d_{\text{eff}}(F)$ and $D(F)$; (c) C_F , $\text{null}(F)$, and ξ_{crit} ; (d) the correlated exposure property; (e) the Fisher–Coalition identity.*

PROOF: Each quantity is defined in terms of $\{s_i\}$ and $\{r_i\}$ only. The aggregate resource function R enters exclusively through the value function v (Definition 2.11), which does not appear in any of the definitions or proofs of (a)–(e). *Q.E.D.*

The Spectral Invariance Theorem is the architectural result that gives the theory its breadth. It establishes a one-directional interaction between two layers: the structural layer—diversity, entropy, vulnerability, adaptive capacity—constrains the economic layer, but the economic layer does not feed back. Any choice of R inherits the complete structural theory without re-derivation. The linear specification is the canonical case in which all gains arise from profile diversity; nonlinear specifications accommodating increasing returns, network effects, and coordination-dependent value leave every structural result intact, and are developed with their domain consequences in the companion paper (Olfati-Saber, 2026b).

6. COLLAPSE OF EXTRACTIVE INSTITUTIONS

The vulnerability apparatus of Section 5 yields its sharpest application in extractive institutions, where sustained extraction drives population profiles toward a common direction and thereby destroys value, resilience, and adaptability simultaneously.

LEMMA 6.1: *If $\mathcal{E} > 1$, then $\phi_P^{\text{actual}} < \phi_P^{\text{Sh}}$.*

PROOF: Budget balance and Shapley efficiency give $\phi_P^{\text{actual}} = \phi_P^{\text{Sh}} - (\phi_E^{\text{actual}} - \phi_E^{\text{Sh}}) < \phi_P^{\text{Sh}}$.
Q.E.D.

COROLLARY 6.2—Blocking Threshold: *The population blocks the actual allocation iff*

$$\mathcal{E} > \mathcal{E}^\circ(F) := \frac{v(F) - v(P)}{\phi_E^{\text{Sh}}} = 1 + \frac{\phi_P^{\text{Sh}} - v(P)}{\phi_E^{\text{Sh}}}. \quad (6.1)$$

PROOF: Budget balance gives $\phi_P^{\text{actual}} = v(F) - \mathcal{E} \cdot \phi_E^{\text{Sh}}$, so $v(P) > \phi_P^{\text{actual}}$ iff $\mathcal{E} > \mathcal{E}^\circ(F)$, dividing by $\phi_E^{\text{Sh}} > 0$ (Corollary 4.6).
Q.E.D.

The blocking threshold $\mathcal{E}^\circ$ is the ratio of the surplus from cooperation with the elites to the elite's fair share: extraction is tolerated only while the elite's contribution justifies it, and collapses once the population would be better off alone.

ASSUMPTION 6.3—Population Homogenization: *Along a sequence of coalition configurations $F(t) = E(t) \cup P(t)$ with fixed group sizes $n_E \ll n_P$ and sustained extraction $\mathcal{E}(t) > 1$, the population's profile density matrix converges to a rank-1 limit: $\rho_{P(t)} \rightarrow s_P s_P^\top$ for some $s_P \in S^{m-1}$.*

The mechanism by which sustained extraction produces population homogenization is a dynamic question, derived in the companion papers on coalition dynamics and extractive institutions (Olfati-Saber, 2026c,e). The present paper establishes the structural consequences: population homogenization alone suffices to destroy value, resilience, and adaptability simultaneously, and these consequences hold regardless of whether the elite maintains or loses its own internal diversity. Throughout, magnitudes are uniformly bounded, $r_i(t) \leq r_{\max}$.

THEOREM 6.4—Structural Fragility of Extractive Institutions: *Under Assumption 6.3 with uniformly bounded magnitudes:*

- (a) *Effective-value collapse.* $\limsup_t d_{\text{eff}}(F(t)) \leq 1 + n_E$. *In the homogenized limit configuration (population profiles equal to s_P), $D(F) \leq 1 + D(E) \leq 1 + n_E$; if $m > 1 + n_E$, the coalition cannot span the capability space regardless of the elite's internal diversity, and $v(F) \leq (1 + n_E) \cdot R(F)$.*
- (b) *Resilience collapse.* *If $m > 1 + n_E$, then at every t there is a subspace $U(t) \subseteq \mathbb{R}^m$ with $\dim U(t) \geq m - 1 - n_E$ such that*

$$\sup_{u \in U(t) \cap S^{m-1}} c_{F(t)}(u) \leq r_{\max} \sum_{i \in P(t)} \|s_i - s_P\|^2 \rightarrow 0, \quad (6.2)$$

so $\xi_{\text{crit}}(F(t)) \rightarrow 0$: the coalition develops near-blind spots in at least $m - 1 - n_E$ independent directions.

- (c) *Adaptive collapse (population).* *The population's within-group profile variance vanishes: $\sigma_{\text{eff}}^2(P(t)) \rightarrow 0$. Since $n_P \gg n_E$, the coalition's aggregate adaptive capacity is dominated by this loss.*
- (d) *Complete collapse.* *If in addition the elite profiles converge to a common direction s_E equal to the population limit, $s_E = s_P = s^*$, then $S(F(t)) \rightarrow 0$ and $\sigma_{\text{eff}}^2(F(t)) \rightarrow 0$.*

PROOF: Assumption 6.3 with fixed n_P gives $\text{tr}(\rho_{P(t)}) - s_P^\top \rho_{P(t)} s_P \rightarrow 0$; equivalently, $(1/n_P) \sum_{i \in P} (1 - (s_i^\top s_P)^2) \rightarrow 0$. Since $1 - x \leq 1 - x^2$ on $[0, 1]$ and $s_i^\top s_P > 0$,

$$\sum_{i \in P(t)} \|s_i - s_P\|^2 = 2 \sum_{i \in P(t)} (1 - s_i^\top s_P) \leq 2n_P (\text{tr}(\rho_{P(t)}) - s_P^\top \rho_{P(t)} s_P) \rightarrow 0, \quad (6.3)$$

so each population profile converges to s_P .

(a). Write $\rho_{F(t)} = \frac{n_P}{n_F} \rho_{P(t)} + \frac{n_E}{n_F} \rho_{E(t)}$. Let $\mathcal{K}$ be the set of matrices of the form $\frac{n_P}{n_F} s_P s_P^\top + \frac{n_E}{n_F} \rho_E$, where ρ_E ranges over the density matrices of n_E -agent elites; $\mathcal{K}$ is compact, and every member has rank at most $1 + n_E$, hence $d_{\text{eff}} \leq 1 + n_E$ on $\mathcal{K}$. By (6.3) the distance from $\rho_{F(t)}$ to $\mathcal{K}$ tends to zero, and d_{eff} is continuous in ρ , so $\limsup_t d_{\text{eff}}(F(t)) \leq \max_{\mathcal{K}} d_{\text{eff}} \leq 1 + n_E$. In the homogenized limit configuration, $\text{span}(F) = \text{span}(s_P) + \text{span}(E)$ and the Grassmann dimension formula gives $D(F) \leq 1 + D(E) \leq 1 + \min(n_E, m)$; the value bound follows from $v = D \cdot R$.

(b). At each t , let $U(t)$ be the orthogonal complement of $\text{span}(\{s_j(t)\}_{j \in E(t)} \cup \{s_P\})$, a subspace of dimension at least $m - 1 - n_E \geq 1$. For a unit vector $u \in U(t)$, every elite term of $c_{F(t)}(u)$ vanishes ($s_j^\top u = 0$), and every population term satisfies $(s_i^\top u)^2 = ((s_i - s_P)^\top u)^2 \leq \|s_i - s_P\|^2$ since $s_P^\top u = 0$. Hence $c_{F(t)}(u) = \sum_{i \in P} r_i (s_i^\top u)^2 \leq r_{\max} \sum_{i \in P} \|s_i - s_P\|^2$, which tends to zero by (6.3), uniformly over $U(t) \cap S^{m-1}$. Since $\xi_{\text{crit}}(F(t)) \leq c_{F(t)}(u)$ for any unit u , the threshold vanishes.

(c). By (6.3), $\sigma_{\text{eff}}^2(P(t)) \leq (1/n_P) \sum_{i \in P} \|s_i - s_P\|^2 \rightarrow 0$ (the variance about the mean is minimal among second moments about any point). The total profile variance decomposes exactly as

$$\sigma_{\text{eff}}^2(F) = \frac{n_P}{n_F} \sigma_{\text{eff}}^2(P) + \frac{n_E}{n_F} \sigma_{\text{eff}}^2(E) + \frac{n_P}{n_F} \|\bar{s}_P - \bar{s}\|^2 + \frac{n_E}{n_F} \|\bar{s}_E - \bar{s}\|^2. \quad (6.4)$$

The first term, weighted by $n_P/n_F \approx 1$, vanishes; the elite's within-group variance is weighted by $n_E/n_F \approx 0$ and cannot compensate.

(d). If $s_E = s_P = s^*$, then $\rho_{F(t)} \rightarrow s^* s^{*\top}$ (rank 1), so $S(F(t)) \rightarrow 0$ by continuity of eigenvalues and entropy, and all four terms in (6.4) vanish. Q.E.D.

COROLLARY 6.5—Vanishing Adaptive Response: *Under the hypotheses of Theorem 6.4(c), the population's expected adaptive response vanishes: $\mathbb{E}_\xi[\Delta \bar{w}_P(t)] \rightarrow 0$. Under part (d), the coalition-wide adaptive response vanishes: $\mathbb{E}_\xi[\Delta \bar{w}(F(t))] \rightarrow 0$.*

PROOF: By the Fisher–Coalition Theorem (Theorem 5.5), $\mathbb{E}_\xi[\Delta \bar{w}] = (\eta\kappa/m) \cdot \sigma_{\text{eff}}^2$, and the relevant variances vanish. Q.E.D.

REMARK 6.6—The carrier of the collapse: Two structural features of Theorem 6.4 deserve emphasis. First, the asymptotic value statement in part (a) is carried by the effective diversity, not by the rank: rank is integer-valued, and along a homogenizing sequence it can remain equal to m at every finite t even as the spectrum concentrates onto a single direction—a rank bound along the sequence would therefore be vacuous exactly where the collapse is underway. The continuous spectral quantity d_{eff} tracks the concentration and obeys the cap $1 + n_E$ in the limit superior; the exact rank cap binds in the homogenized limit configuration. Second, part (b) is uniform in the elite: the near-blind subspace $U(t)$ is constructed fresh at each t from whatever the elite profiles then are, so the resilience collapse holds even if the elite continually reorients. Both features are verified numerically in the Appendix, where rank remains maximal while d_{eff} falls to within a few percent of one.

Population homogenization alone thus suffices to destroy all three channels through which diversity sustains a coalition: it caps the value the coalition can create, it opens blind spots to environmental shocks, and it extinguishes the variance on which adaptive selection depends. Crucially, these consequences do not require the elite to homogenize: a small elite embedded in a high-dimensional environment cannot compensate for the loss of population diversity. The coalition is not merely fragile; it is frozen—unable to create value, unable to withstand shocks, unable to adapt.

The Structural Fragility Theorem holds for any aggregate resource function R , by the Spectral Invariance Theorem: the collapse threshold, null space, and adaptive capacity are structural quantities that do not depend on R . What R changes is the stake: under increasing returns, the value destroyed when an exposed direction is struck is larger, and the extractable surplus sustaining $\mathcal{E} > 1$ is larger—making extraction more tempting precisely where its structural consequences are most costly.

7. INTERNATIONAL TRADE

We interpret agents as countries and $q_i \in \mathbb{R}_{++}^m$ as country i 's productive capacity across m sectors. The value function $v(F) = D(F) \cdot \sum_{i \in F} r_i$ then assigns to a trading bloc its rank diversity times its total productive capacity, and the gains from trade are $(D(F) - 1) \cdot \sum r_i$ —positive exactly when the bloc spans more than one capability direction, when its members have different comparative advantages. Profile non-parallelism provides a geometric unification of the two classical trade theories.

PROPOSITION 7.1—Ricardian Comparative Advantage: *In the two-good Ricardian model with labor productivities $(a_{i1}, a_{i2}) \in \mathbb{R}_{++}^2$, countries i and j have different comparative advantage iff $s_i \neq s_j$, iff $a_{i1}/a_{i2} \neq a_{j1}/a_{j2}$.*

PROOF: $s_i \parallel s_j$ iff $(a_{i1}, a_{i2}) = \alpha(a_{j1}, a_{j2})$ for some $\alpha > 0$, iff $a_{i1}/a_{i2} = a_{j1}/a_{j2}$. Q.E.D.

PROPOSITION 7.2—Heckscher–Ohlin as Profile Geometry: *With factor endowments $f_i = (K_i, L_i)^\top \in \mathbb{R}_{++}^2$ and rank-2 technology matrix $A \in \mathbb{R}_+^{m \times 2}$ mapping factors to goods, $s_i \parallel s_j$ iff $K_i/L_i = K_j/L_j$.*

PROOF: $Af_i \parallel Af_j$ iff $A(f_i - \alpha f_j) = 0$ for some $\alpha > 0$. Since $\text{rank}(A) = 2$, A is injective, so $f_i = \alpha f_j$. Q.E.D.

COROLLARY 7.3—Gains from Trade: *Under $v(F) = D(F) \cdot \sum_{i \in F} r_i$, the gains from trade are*

$$G(F) = (D(F) - 1) \cdot \sum_{i \in F} r_i, \quad (7.1)$$

positive iff $D(F) \geq 2$.

PROOF: Since $v(\{i\}) = r_i$, the result follows by subtraction. Q.E.D.

Ricardian comparative advantage is a statement about productivity ratios; Heckscher–Ohlin comparative advantage is a statement about capital–labor ratios. In the present framework, both are the same condition— $s_i \neq s_j$, distinct capability profiles—expressed in the coordinates natural to each theory. The gains from trade are proportional to the rank diversity of the trading bloc, and the Superadditivity Theorem guarantees that a bloc of countries with complementary profiles gains strictly from trade, with the cross terms pricing the surplus from complementarity.

REMARK 7.4: The corollary gives $G(F) = 0$ when $D(F) = 1$, which does not capture gains from trade between identical economies—the central insight of [Krugman \(1979\)](#). The generalized value function $v(F) = D(F) \cdot R(F)$ accommodates such settings through nonlinear R ; by Theorem 5.6 the structural results are unaffected. The companion paper ([Olfati-Saber, 2026b](#)) develops increasing returns to variety, network effects, and concave returns as nonlinear specializations.

8. THE TRADITIONS RE-EXAMINED, AND PREDICTIONS

8.1. Four Traditions

Cooperative game theory. The classical theory is not replaced; it is relocated. The Shapley value, the core, balancedness, and convexity are results about allocation given v , and they survive intact one level up: the present framework derives the v they operate on. What changes is the epistemic status of the characteristic function—from primitive to theorem—and with it the reach of the allocation results: the Shapley Decomposition (Theorem 4.5) shows what the Shapley value *means* under the derived v , resolving each payoff into a return on magnitude and a rent on position, and the generic-position closed forms (Proposition 4.7) show how those components behave as coalitions scale. Coalition structures in the sense of [Aumann and Drèze \(1974\)](#) acquire a geometry: a partition of $\mathcal{A}$ is a partition of profile space.

Collective-intelligence empirics. [Hong and Page \(2004\)](#) proved that diverse groups of problem solvers can outperform groups of high-ability problem solvers, and [Woolley et al. \(2010\)](#) identified an empirical collective-intelligence factor in group performance. Both are landmark results, and both lacked a formal measure of collective intelligence as a function of the group’s structural properties. Definition 2.10 supplies the measure: the diversity-trumps-ability phenomenon is, in the present language, the statement that rank diversity can outweigh magnitude in $v = D \cdot R$ —span beats scale when the span gap is large enough—and the c -factor is the empirical shadow of $CI(F) = d_{\text{eff}} \cdot C \cdot \mathcal{F}$, three bottlenecks that the theory says should each independently predict performance.

Institutional economics. The inclusive–extractive dichotomy of [Acemoglu and Robinson \(2012\)](#), the institutional persistence of [North \(1990\)](#), the collective-action logic of [Olson \(1965\)](#), and the self-governance results of [Ostrom \(1990\)](#) all concern how allocation rules feed back on what a society can do. Definition 2.13 formalizes extraction as a ratio to the fair Shapley share, the Blocking Threshold (Corollary 6.2) prices its tolerance, and the Structural Fragility Theorem derives the feedback’s endpoint: extraction that homogenizes destroys the three spectral capacities at once. The dynamic mechanism—why extraction homogenizes—is a theorem of the companion papers ([Olfati-Saber, 2026c,e](#)).

Trade theory. Ricardian comparative advantage ([Ricardo, 1817](#)), factor-endowment trade ([Heckscher, 1919](#), [Ohlin, 1933](#)), and the continuum Ricardian model ([Dornbusch, Fischer and Samuelson, 1977](#)) are unified by profile geometry: different theories of *why* profiles differ, one theorem about *what* profile difference is worth. The modern trade theory gains between identical economies ([Krugman, 1979](#), [Helpman and Krugman, 1985](#)) live in the domain layer, as nonlinear R ([Olfati-Saber, 2026b](#)).

8.2. Predictions

The theory generates falsifiable predictions. Each is stated with its generating result and the observation that would disconfirm it.

Prediction 1 (span-expansion premium). In teams with measurable skill profiles, the value added by a new hire decomposes into a magnitude term and a span term, and the span term is

earned only by hires whose profile expands the team’s span (Theorems 4.2 and 4.5). Controlling for magnitude, span-expanding hires should command a premium. *Disconfirming observation*: hire value tracking magnitude alone, with no premium for span expansion at fixed magnitude.

Prediction 2 (dissipation of specialist rents). The span premium declines as $1/n$ once team size exceeds the effective skill dimension (Corollary 4.8): rare-specialist premia should be largest in small or dimension-starved teams and fall toward pure magnitude pricing in large ones. *Disconfirming observation*: size-invariant specialist premia in large organizations whose skill profiles are in generic position.

Prediction 3 (diversity versus ability). Group performance is ordered by $D \cdot R$, not by R : a group with lower total ability but strictly larger span outperforms when the rank advantage exceeds the magnitude ratio (Definition 2.11, Theorem 4.3). *Disconfirming observation*: performance ordering following total ability at fixed span across the comparison.

Prediction 4 (spectral fragility). Losses from novel shocks—shocks orthogonal to a group’s dominant capability directions—are predicted by pre-shock spectral concentration: at fixed size and resources, groups with smaller $\lambda_{\min}(C_F)$ suffer disproportionately, with the entropy bound (5.1) supplying the scaling near homogeneity (Theorem 5.3). *Disconfirming observation*: shock losses uncorrelated with pre-shock spectral concentration after conditioning on size and resources.

Prediction 5 (adaptation proportional to variance). Post-shock improvement is proportional to pre-shock profile variance, with the same proportionality constant across shock directions on average (Theorem 5.5). *Disconfirming observation*: adaptive recovery uncorrelated with pre-shock profile variance, or systematically dependent on shock direction after averaging.

Prediction 6 (blocking and the triple collapse). Exit, secession, or institutional rupture occurs when the measured extraction ratio crosses the blocking threshold (6.1); and sustained extraction is accompanied by declining population-level profile variance, with value, resilience, and adaptability deteriorating together rather than separately (Corollary 6.2, Theorem 6.4). *Disconfirming observation*: stable cooperation under persistent supra-threshold extraction, or sustained deep extraction coexisting with stable population-level profile diversity.

Prediction 7 (trade gains track bloc rank). At fixed total productive capacity, gains from trade are ordered by the rank diversity of the bloc—by how many independent capability directions its members span—not by bloc size (Corollary 7.3). *Disconfirming observation*: trade gains tracking bloc size or capacity alone, with no residual explained by profile rank.

9. CONCLUSION

We have developed Coalition Intelligence Theory, a structural theory of heterogeneous interacting agents that replaces the unstructured characteristic function of cooperative game theory with the factored decomposition $v(F) = D(F) \cdot R(F)$, grounded in the capabilities of the coalition’s members. The framework resolves the three problems that classical theory left open: the construction of a value function from primitives, the diagnosis of structural vulnerability, and the decomposition of the Shapley value into channels traceable to coalition structure.

The central object is the profile density matrix, whose spectrum encodes three mathematically distinct roles of diversity. Rank diversity drives value creation through the Superadditivity Theorem. The null space and collapse threshold determine resilience: the coalition is invulnerable to shocks from all directions if and only if $D(F) = m$, and near homogeneity the threshold is controlled linearly by the entropy. The effective profile variance governs adaptive capacity through the Fisher–Coalition Theorem, vanishing if and only if the coalition is homogeneous. These are independent bottlenecks: a coalition can fail through any one while succeeding on the others. The Shapley Decomposition and its generic-position closed forms make the allocation prescriptive—a rank-weighted base identical across agents, and a span-expansion bonus

that is a scarcity rent of position, dissipating as the coalition outgrows its capability dimension. And the two applications show the apparatus at work: extraction-driven homogenization destroys value, resilience, and adaptability simultaneously, and profile non-parallelism unifies the classical trade theories.

The Spectral Invariance Theorem gives the theory its architecture: a structural layer invariant to R , and a domain layer that is field-specific and interchangeable. The framework thereby transforms cooperative game theory from descriptive to prescriptive: the decomposition tells you whom to add (span-expanding agents), when formation is worthwhile ($D \geq 2$), where the vulnerabilities lie (the null space), and whether the allocation is fair and stable. The hiring problem, the trade problem, the institutional stability problem, and the organizational design problem are the same problem viewed through different specializations of R —and the domain layers extend beyond economics, to cell populations as cytodynamics in the biological companions of this series.

Open problems remain. A complete theory of core stability under $v = D \cdot R$ requires characterizing the profile configurations under which the game is convex; partial results on blocked inclusion under extraction appear in the institutional companion (Olfati-Saber, 2026e). Measurement protocols for profiles, coordination capacity, and signal fidelity are needed to carry the predictions of Section 8.2 to data. Endogenous network formation would connect the framework to strategic network theory (Jackson, 2008). The companion papers develop the remaining layers: the decomposition and nonlinear specifications (Olfati-Saber, 2026b), the dynamics of the spectrum (Olfati-Saber, 2026c), and the evolutionary theory in which demography prices the maintenance of diversity (Olfati-Saber, 2026f).

APPENDIX A: NOTATION

Symbol	Meaning
$q_i = r_i s_i$	capability vector; magnitude r_i , profile $s_i \in S^{m-1}$
m	dimension of capability space
F, n_F	coalition and its size
$D(F)$	rank diversity, $\text{rank}(S_F)$
ρ_F	profile density matrix, $(1/n_F) \sum_i s_i s_i^\top$
$S(F), d_{\text{eff}}(F)$	von Neumann entropy; effective diversity e^S
$C(F), \mathcal{F}(F)$	coordination capacity; signal fidelity
$\text{CI}(F)$	collective intelligence, $d_{\text{eff}} \cdot C \cdot \mathcal{F}$
$R(F), v(F)$	aggregate resource function; value $D \cdot R$
$\phi_i^{\text{Sh}}, \gamma_i, \beta_i$	Shapley value; rank-weighted base coefficient; span bonus
$C_F, c_F(u)$	capability matrix $\sum_i r_i s_i s_i^\top$; directional capability
$\xi_{\text{crit}}(F)$	collapse threshold, $\lambda_{\min}(C_F)$
$\sigma_{\text{eff}}^2(F)$	profile variance, $(1/n_F) \sum_i \ s_i - \bar{s}\ ^2$
$E, P, \mathcal{E}, \mathcal{E}^\circ$	elites; masses; extraction ratio; blocking threshold
K	$\min(m-1, n-1)$ in the generic-position closed forms

APPENDIX B: PROOF OF PROPOSITION 4.7

Generic-position consequences. Since every subset of at most m profiles is linearly independent, two simplifications hold for every $S \subseteq F$. First, the rank of S equals its size until the ambient dimension is saturated:

$$D(S) = \min(|S|, m). \quad (\text{B.1})$$

Second, agent i 's profile s_i lies outside $\text{span}(S)$ if and only if $|S| \leq m - 1$: when $|S| \leq m - 1$, the set $S \cup \{i\}$ has at most m elements, so its profiles are linearly independent and $s_i \notin \text{span}(S)$; when $|S| \geq m$, the profiles of S already span $\mathbb{R}^m$, so $s_i \in \text{span}(S)$.

Part (a): the base coefficient γ_i . Under a uniform random ordering π , agent i is equally likely to occupy any of the n positions, so the predecessor count $|S_\pi|$ is uniform on $\{0, 1, \dots, n - 1\}$. From the decomposition in Theorem 4.5,

$$\gamma_i = \mathbb{E}_\pi[D(S_\pi \cup \{i\})] = \frac{1}{n} \sum_{k=0}^{n-1} [\min(k, m) + \mathbf{1}[k \leq m - 1]]. \quad (\text{B.2})$$

For $k \leq m - 1$ the summand is $k + 1$; for $k \geq m$ it is m . Reindexing with $j = k + 1$,

$$\gamma_i = \frac{1}{n} \sum_{j=1}^n \min(j, m). \quad (\text{B.3})$$

Case $n \leq m$. Every $j \leq n \leq m$, so $\min(j, m) = j$ and

$$\gamma_i = \frac{1}{n} \sum_{j=1}^n j = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}. \quad (\text{B.4})$$

Case $n \geq m$. The sum splits at $j = m$:

$$\sum_{j=1}^n \min(j, m) = \underbrace{\sum_{j=1}^m j}_{\text{rising part}} + \underbrace{\sum_{j=m+1}^n m}_{\text{flat part}} = \frac{m(m+1)}{2} + (n-m)m. \quad (\text{B.5})$$

Dividing by n ,

$$\gamma_i = \frac{1}{n} \left[\frac{m(m+1)}{2} + nm - m^2 \right] = \frac{1}{n} \left[nm - \frac{m(m-1)}{2} \right] = m - \frac{m(m-1)}{2n}. \quad (\text{B.6})$$

In either case, γ_i depends only on n and m , not on the identity of agent i , so the base coefficient is the same for every agent.

Part (b): the span-expansion bonus β_i . From the decomposition,

$$\beta_i = \mathbb{E}_\pi[\mathbf{1}[s_i \notin \text{span}(S_\pi)] \cdot R_{S_\pi}]. \quad (\text{B.7})$$

The indicator restricts to $|S_\pi| \leq m - 1$, i.e., $|S_\pi| \leq K$ where $K = \min(m - 1, n - 1)$. Conditioning on $|S_\pi| = k$ and using $\mathbb{P}(|S_\pi| = k) = 1/n$,

$$\beta_i = \sum_{k=0}^K \frac{1}{n} \mathbb{E}[R_{S_\pi} \mid |S_\pi| = k]. \quad (\text{B.8})$$

Given $|S_\pi| = k$, the predecessor set is a uniformly random k -subset of $F \setminus \{i\}$. By symmetry, each of the $n - 1$ agents in $F \setminus \{i\}$ appears in this k -subset with probability $k/(n - 1)$, so

$$\mathbb{E}[R_{S_\pi} \mid |S_\pi| = k] = \sum_{j \neq i} r_j \cdot \frac{k}{n-1} = \frac{k(R_F - r_i)}{n-1}. \quad (\text{B.9})$$

Substituting,

$$\beta_i = \frac{R_F - r_i}{n(n-1)} \sum_{k=0}^K k = \frac{R_F - r_i}{n(n-1)} \cdot \frac{K(K+1)}{2} = (R_F - r_i) \frac{K(K+1)}{2n(n-1)}. \quad (\text{B.10})$$

Part (c): the efficiency identity. We verify $\sum_{i \in F} \phi_i^{\text{Sh}} = v(F) = D(F) \cdot R_F$ in each regime. Since $\gamma_i = \gamma$ is constant across agents, $\sum_i \gamma_i r_i = \gamma R_F$. For the bonus term, $\sum_i (R_F - r_i) = nR_F - R_F = (n-1)R_F$, so

$$\sum_i \beta_i = \frac{K(K+1)}{2n(n-1)} \cdot (n-1) R_F = \frac{K(K+1)}{2n} R_F. \quad (\text{B.11})$$

Case $n \geq m$ ($D(F) = m$, $K = m-1$):

$$\sum_i \phi_i = \left[m - \frac{m(m-1)}{2n} \right] R_F + \frac{(m-1)m}{2n} R_F = m R_F = D(F) R_F. \quad (\text{B.12})$$

Case $n \leq m$ ($D(F) = n$, $K = n-1$):

$$\sum_i \phi_i = \frac{n+1}{2} R_F + \frac{(n-1)n}{2n} R_F = \frac{n+1}{2} R_F + \frac{n-1}{2} R_F = n R_F = D(F) R_F. \quad (\text{B.13})$$

In both cases, the base and bonus terms cancel exactly to recover $v(F)$.

Q.E.D.

APPENDIX: REFERENCES

- Acemoglu, Daron, and James A. Robinson (2012), *Why Nations Fail*. Crown Business, New York. [16]
- Arkoulakis, Costas, Arnaud Costinot, and Andrés Rodríguez-Clare (2012), “New Trade Models, Same Old Gains?” *American Economic Review*, 102 (1), 94–130. []
- Aumann, Robert J., and Jacques H. Drèze (1974), “Cooperative Games with Coalition Structures.” *International Journal of Game Theory*, 3 (4), 217–237. [16]
- Bondareva, Olga N. (1963), “Some Applications of Linear Programming Methods to the Theory of Cooperative Games.” *Problemy Kibernetiki*, 10, 119–139. [1]
- Cover, Thomas M., and Joy A. Thomas (2006), *Elements of Information Theory* (2nd ed.). Wiley, Hoboken, NJ. []
- Dornbusch, Rudiger, Stanley Fischer, and Paul A. Samuelson (1977), “Comparative Advantage, Trade, and Payments in a Ricardian Model with a Continuum of Goods.” *American Economic Review*, 67 (5), 823–839. [16]
- Fiedler, Miroslav (1973), “Algebraic Connectivity of Graphs.” *Czechoslovak Mathematical Journal*, 23 (2), 298–305. [4]
- Fisher, Ronald A. (1930), *The Genetical Theory of Natural Selection*. Clarendon Press, Oxford. []
- Heckscher, Eli (1919), “The Effect of Foreign Trade on the Distribution of Income.” *Ekonomisk Tidskrift*, 21, 497–512. [16]
- Helpman, Elhanan, and Paul R. Krugman (1985), *Market Structure and Foreign Trade*. MIT Press, Cambridge, MA. [16]
- Hong, Lu, and Scott E. Page (2004), “Groups of Diverse Problem Solvers Can Outperform Groups of High-Ability Problem Solvers.” *Proceedings of the National Academy of Sciences*, 101 (46), 16385–16389. [16]
- Jackson, Matthew O. (2008), *Social and Economic Networks*. Princeton University Press, Princeton, NJ. [18]
- Krugman, Paul R. (1979), “Increasing Returns, Monopolistic Competition, and International Trade.” *Journal of International Economics*, 9 (4), 469–479. [16]
- MacArthur, Robert H. (1955), “Fluctuations of Animal Populations and a Measure of Community Stability.” *Ecology*, 36 (3), 533–536. []
- Nash, John F. (1950), “Equilibrium Points in n -Person Games.” *Proceedings of the National Academy of Sciences*, 36 (1), 48–49. []

- North, Douglass C. (1990), *Institutions, Institutional Change, and Economic Performance*. Cambridge University Press, Cambridge, U.K. [16]
- Ohlin, Bertil (1933), *Interregional and International Trade*. Harvard University Press, Cambridge, MA. [16]
- Olfati-Saber, Reza (2026b), “Decomposition of the Value Function: Coalition Intelligence Theory and the Structural Foundations of Cooperative Games.” Working paper, Institute for Intelligence Science and Society. [2, 12, 16, 18]
- Olfati-Saber, Reza (2026c), “Coalition Dynamics: Spectral Evolution of Collective Intelligence.” Working paper, Institute for Intelligence Science and Society. [2, 13, 16, 18]
- Olfati-Saber, Reza (2026e), “The Rise and Fall of Extractive Institutions: A Coalition-Theoretic Analysis.” Working paper, Institute for Intelligence Science and Society. [2, 13, 16, 18]
- Olfati-Saber, Reza (2026f), “Evolutionary Coalition Dynamics: Demography, Variance Balance, and the Critical Mutation Rate.” Working paper, Institute for Intelligence Science and Society. [2, 18]
- Olson, Mancur (1965), *The Logic of Collective Action*. Harvard University Press, Cambridge, MA. [16]
- Ostrom, Elinor (1990), *Governing the Commons*. Cambridge University Press, Cambridge, U.K. [16]
- Ricardo, David (1817), *On the Principles of Political Economy and Taxation*. John Murray, London. [16]
- Shannon, Claude E. (1948), “A Mathematical Theory of Communication.” *Bell System Technical Journal*, 27 (3), 379–423. []
- Shapley, Lloyd S. (1953), “A Value for n -Person Games.” In *Contributions to the Theory of Games II* (Harold W. Kuhn and Albert W. Tucker, eds.), 307–317, Princeton University Press, Princeton, NJ. [1, 7]
- Shapley, Lloyd S. (1967), “On Balanced Sets and Cores.” *Naval Research Logistics Quarterly*, 14 (4), 453–460. [1]
- Tilly, Charles (1998), *Durable Inequality*. University of California Press, Berkeley. []
- von Neumann, John (1927), “Thermodynamik quantenmechanischer Gesamtheiten.” *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen*, 273–291. []
- von Neumann, John, and Oskar Morgenstern (1944), *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ. [1]
- Woolley, Anita Williams, Christopher F. Chabris, Alex Pentland, Nada Hashmi, and Thomas W. Malone (2010), “Evidence for a Collective Intelligence Factor in the Performance of Human Groups.” *Science*, 330 (6004), 686–688. [16]